Stirling's Formula

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Following preliminary work by de Moivre (1718) Stirling in 1730 [4] stated his famous formula that expresses the factorial in a way that leads to a very useful assessment of its asymptotic behaviour. Here we reproduce the notably narrow bounds given by Robbins [3] following a method attributed to Cesàro [1] and Fisher [2].

Theorem 1 *For all natural numbers $n \geq 1$ we have*

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot e^{r_n}$$

where the error term r_n is bounded by

$$\frac{1}{12n+1} \leq r_n \leq \frac{1}{12n}$$

The approximation is illustrated by the following table, where s_n is the upper bound and t_n, the lower bound from the theorem.

n	1	2	3	4	5	6	7	8	9
s_n	1.002	2.001	6.001	24.001	120.003	720.01	5040.04	40320.2	362881.4
$n!$	1	2	6	24	120	720	5040	40320	362880
t_n	0.996	1.997	5.996	23.991	119.970	719.87	5039.33	40315.9	362850.1

This suggests that the upper bound is closer to the true value then the lower bound; and the absolute errors increase. The relative errors however decrease quite fast, see Corollary 1 below.

Proof. We consider the sequence

$$a_n = \frac{n!}{(\frac{n}{e})^n \cdot \sqrt{n}}$$

and show that it decreases monotonically; because all of its members are positive, we then know that it converges.

Dividing two consecutive terms we get

$$\frac{a_n}{a_{n+1}} = \frac{n!(\frac{n+1}{e})^{n+1} \cdot \sqrt{n+1}}{(\frac{n}{e})^n \cdot \sqrt{n} \cdot (n+1)!} = \frac{1}{e} \cdot (\frac{n+1}{n})^{n+1/2},$$

$$\log \frac{a_n}{a_{n+1}} = -1 + (n + \frac{1}{2}) \cdot \log \frac{n+1}{n}.$$

Lemma 2 below immediately gives

$$0 < \frac{1}{12} \cdot (\frac{1}{n + \frac{1}{12}} - \frac{1}{n + \frac{1}{12} + 1}) < \log \frac{a_n}{a_{n+1}} < \frac{1}{12} \cdot (\frac{1}{n} - \frac{1}{n+1}).$$

From the left inequality we conclude $a_n > a_{n+1}$ as claimed.

Now let $a = \lim_{n\to\infty} a_n$. Then $a \geq 0$ and by telescoping

$$\frac{1}{12} \cdot (\frac{1}{n + \frac{1}{12}} - \frac{1}{n + \frac{1}{12} + k}) < \log \frac{a_n}{a_{n+k}} < \frac{1}{12} \cdot (\frac{1}{n} - \frac{1}{n+k}).$$

For $k \to \infty$ we get

$$\frac{1}{12n+1} \leq \log \frac{a_n}{a} \leq \frac{1}{12n},$$

$$e^{\frac{1}{12n+1}} \leq \frac{a_n}{a} \leq e^{\frac{1}{12n}}.$$

To complete the proof of the theorem we have to show that $a = \sqrt{2\pi}$.

From Wallis' product formula, see Lemma 3 below, and using $k! = a_k k^{k+1/2}/e^k$, we get

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{a_n^2 \cdot n^{2n+1} \cdot 2^{2n} \cdot e^{2n}}{e^{2n} \cdot a_{2n} \cdot (2n)^{2n+1/2} \cdot \sqrt{n+1/2}}$$

$$= a \cdot \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2} \cdot \sqrt{n+1/2}} = \frac{a}{\sqrt{2}}.$$

Therefore $a = \sqrt{2\pi}$. $\diamond$

Lemma 1 *For* $0 < x < 1$

$$\frac{3x}{3-x^2} < \frac{1}{2} \log \frac{1+x}{1-x} < x \cdot \left(1 + \frac{1}{3} \cdot \frac{x^2}{1-x^2}\right).$$

Proof. For $|x| < 1$ we have the well-known power series expansion

$$\frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \sum_{\nu=1}^{\infty} \frac{x^{2\nu-1}}{2\nu-1}.$$

For $0 < x < 1$ we get the upper bound

$$\frac{1}{2} \log \frac{1+x}{1-x} < x + \frac{x^3}{3} + \frac{x^5}{3} \cdots = x + \sum_{\nu=2}^{\infty} \frac{x^{2\nu-1}}{3} = x + \frac{x^3}{3} \left(1 + x^2 + x^4 + \cdots\right)$$

$$= x + \frac{x^3}{3} \cdot \frac{1}{1-x^2} = x \cdot \left(1 + \frac{1}{3} \cdot \frac{x^2}{1-x^2}\right).$$

For the lower bound we use

$$\frac{1}{2}\log\frac{1+x}{1-x} > x + \frac{x^3}{3} + \frac{x^5}{9}\cdots = \sum_{\nu=1}^{\infty}\frac{x^{2\nu-1}}{3^{\nu-1}} = x\cdot\sum_{\nu=0}^{\infty}\frac{x^{2\nu}}{3^{\nu}} = x\cdot\frac{1}{1-\frac{x^2}{3}}\,.$$

$\diamond$

Lemma 2 *For* $n \in \mathbb{N}_1$

$$2+\frac{1}{6}\cdot\left(\frac{1}{n+\frac{1}{12}}-\frac{1}{n+\frac{1}{12}+1}\right) < (2n+1)\cdot\log\frac{n+1}{n} < 2+\frac{1}{6}\cdot\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Proof. In Lemma 1 we substitute $x = \frac{1}{2n+1}$. Then

$$\frac{1+x}{1-x} = \frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} = \frac{2n+2}{2n} = \frac{n+1}{n}.$$

This gives the upper bound

$$\frac{1}{2}\cdot\log\frac{n+1}{n} < \frac{1}{2n+1}\cdot\left(1+\frac{1}{3}\cdot\frac{1}{4n^2+4n}\right) = \frac{1}{2n+1}\cdot\left(1+\frac{1}{12}\cdot\frac{1}{n(n+1)}\right),$$

as claimed. At the lower bound we get

$$\frac{1}{2}\cdot\log\frac{n+1}{n} > \frac{3(2n+1)}{3(2n+1)^2-1},$$

whence

$$(2n+1)\cdot\log\frac{n+1}{n} > \frac{6(2n+1)^2}{3(2n+1)^2-1} = 2+\frac{2}{3(2n+1)^2-1} = 2+\frac{2}{12n^2+12n+2}\,.$$

The lower bound we aim at evaluates to

$$2+\frac{1}{6}\cdot\left(\frac{1}{n+\frac{1}{12}}-\frac{1}{n+\frac{1}{12}+1}\right) = 2+2\cdot\left(\frac{1}{12n+1}-\frac{1}{12n+13}\right)$$

$$= 2+2\cdot\frac{12}{(12n+1)(12n+13)} = 2+2\cdot\frac{12}{12\cdot 12n^2+14\cdot 12n+13} = 2+2\cdot\frac{2}{12n^2+14n+\frac{13}{12}}$$

which is clearly smaller for $n \geq 1$. $\diamond$

Lemma 3 (Product formula of Wallis**)**

$$\sqrt{\pi} = \lim_{n\to\infty}\frac{2^{2n}\cdot(n!)^2}{(2n)!\cdot\sqrt{n+1/2}}.$$

Proof. Starting with the product expansion of the sine function,

$$\sin(\pi x) = \pi x \cdot \prod_{k=1}^{\infty}(1 - \frac{x^2}{k^2}),$$

and substituting $x = 1/2$, we get

$$1 = \frac{\pi}{2} \cdot \prod_{k=1}^{\infty} \frac{4k^2 - 1}{4k^2},$$

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{(2k)^4}{(2k-1)2k \cdot 2k(2k+1)} = \lim_{n\to\infty} \frac{2^{4n} \cdot (n!)^4}{((2n)!)^2(2n+1)},$$

and this immediately gives the assertion. $\diamond$

Corollary 1 *If we replace $n!$ by $s_n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot e^{\frac{1}{12n}}$, the relative error is bounded by*

$$1 \leq \frac{s_n}{n!} < e^{\frac{1}{(12n)^2}}.$$

Proof. Let $t_n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot e^{\frac{1}{12n+1}}$. Then

$$1 \leq \frac{s_n}{n!} \leq \frac{s_n}{t_n} = e^{\frac{1}{12n} - \frac{1}{12n+1}} = e^{\frac{1}{12n(12n+1)}} < e^{\frac{1}{(12n)^2}}.$$

$\diamond$

Note that the "usual" textbook estimate gives the lower bound $1 \leq r_n$. From this we get the bound $e^{\frac{1}{12n}}$ for the relative error that has only a linear term in the denominator of the exponential instead of the quadratic one.

Corollary 2 *For all natural numbers $n \geq 1$*

$$\frac{n!\, e^n}{n^n} = \sqrt{2\pi n} \cdot u_n$$

where the error term u_n is bounded by

$$1 + \frac{1}{13n} < u_n < 1 + \frac{1}{11n}.$$

For $n \to \infty$

$$\frac{n!\, e^n}{n^n} = \sqrt{2\pi n} + \mathrm{O}(\frac{1}{\sqrt{n}}).$$

Proof. We use the inequality $e^x > 1 + x$ for all real $x \neq 0$. For $0 < x < 1$ we therefore have $1 - x \leq e^{-x}$, whence $e^x \leq \frac{1}{1-x} = 1 + \frac{1}{\frac{1}{x}-1}$. Therefore

$$\frac{n!\, e^n}{n^n} < \sqrt{2\pi n} \cdot \left(1 + \frac{1}{12n-1}\right) \leq \sqrt{2\pi n} \cdot \left(1 + \frac{1}{11n}\right).$$

For the lower bound we have

$$\frac{n!\, e^n}{n^n} > \sqrt{2\pi n} \cdot \left(1 + \frac{1}{12n+1}\right) \geq \sqrt{2\pi n} \cdot \left(1 + \frac{1}{13n}\right).$$

$\diamond$

Corollary 3 *For all natural numbers $n \geq 1$*

$$\frac{n^n}{n!\, e^n} = \frac{1}{\sqrt{2\pi n}} \cdot v_n$$

where the error term v_n is bounded by

$$1 - \frac{1}{12n} < v_n < 1 - \frac{1}{14n}.$$

For $n \to \infty$

$$\frac{n^n}{n!\, e^n} = \frac{1}{\sqrt{2\pi n}} + \mathrm{O}(\frac{1}{\sqrt{n^3}}).$$

Proof. The lower bound is immediate from $1 - x \leq e^{-x}$. For the upper bound we use $e^{-x} < \frac{1}{1+x} = 1 - \frac{1}{\frac{1}{x}+1}$, and get

$$\frac{n^n}{n!\, e^n} < \frac{1}{\sqrt{2\pi n}} \cdot \left(1 - \frac{1}{12n+2}\right) \leq \frac{1}{\sqrt{2\pi n}} \cdot \left(1 - \frac{1}{14n}\right).$$

$\diamond$

Applying the theorem to the middle binomial coefficients $\binom{2n}{n}$ yields:

Corollary 4

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{4^n}{\sqrt{\pi n}} \cdot w_n,$$

where the error term w_n is bounded by

$$e^{-\frac{1}{6n}} < w_n < 1.$$

Proof. Intermediate steps:

$$\frac{(2n)!}{(n!)^2} = \frac{\sqrt{4\pi n} \cdot \left(\frac{2n}{e}\right)^{2n} \cdot e^{r_{2n}}}{2\pi n \cdot \left(\frac{n}{e}\right)^{2n} \cdot e^{2r_n}}$$

and

$$\frac{1}{30n} < r_{2n} \leq \frac{1}{24n} < \frac{1}{12n}, \quad \frac{1}{12n} < 2r_n \leq \frac{1}{6n},$$

$$e^{-\frac{1}{6n}} < e^{[-\frac{4}{30}]\cdot\frac{1}{n}} = e^{[\frac{1}{30}-\frac{1}{6}]\cdot\frac{1}{n}} < w_n = \frac{e^{r_{2n}}}{e^{2r_n}} < 1.$$

$\diamond$

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 1$ be the Catalan numbers.

Corollary 5

$$C_n = \frac{4^n}{(n+1)\sqrt{\pi n}} \cdot w_n,$$

with w_n as in Corollary 4.

References

[1] E. Cesàro: *Corso di analisi algebrica con introduzione al calcolo infinitesimale.* Bocca, Torino 1894.

[2] A. Fisher: *Mathematical Theory of Probabilities.* Macmillan, New York 1915.

[3] H. Robbins: A remark on Stirling's formula. Amer. Math. Monthly 62 (1955), 26–29.

[4] J. Stirling: *Methodus Differentialis: sive Tractatus de Summatione et Interpolatione Serierum Infinitarum.* G. Strahan, Londini (London) 1730.