On Stabilizers and Orbits of Binary Forms

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This text contains some elementary, essentially trivial, and nasty, but useful calculations for the standard actions of the special linear group of order 2 and its Lie algebra over an arbitrary field. We determine the stabilizers of some sample binary forms and discuss the closedness of their orbits, as well as the separability of their orbit maps. For an application to invariant theory see [4] or [2]. Some of the results were stated in [2] without proofs.

The determination of the stabilizers uses only elementary algebra—that is the solution of equations—and is valid over any field (with some variations in the results). The statements on orbits assume that the base field is algebraically closed. Their proofs use some facts from algebraic geometry, in particular on algebraic goups and their actions. For these we refer to [1] and [5].

1 The Operation of the Group SL_2 and its Lie Algebra $\mathfrak{sl}_2$

1.1 The action of SL_2 on binary forms

Let k be a field. We consider the group $G = SL_2(k)$ of 2×2-matrices with determinant 1 over k. The matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \tag{1}$$

acts on the 2-dimensional vector space k^2 by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Denote the coordinate functions $k^2 \longrightarrow k$ by X and Y, where

$$X \begin{pmatrix} x \\ y \end{pmatrix} = x, \quad Y \begin{pmatrix} x \\ y \end{pmatrix} = y$$

for all $x, y \in k$. Since the inverse of g is

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

the induced ("contragredient") action on the space of linear forms spanned by the coordinate functions X and Y is given by

$$\begin{aligned} X &\mapsto dX - bY, \\ Y &\mapsto -cX + aY. \end{aligned}$$

(In general a function $f\colon k^2 \longrightarrow k$ is transformed to $f \circ g^{-1}$.) This action extends to the polynomial ring $k[X, Y]$ as group of automorphisms. In particular for the powers of the coordinate functions we get the formulas

$$\begin{aligned} X^r \mapsto (dX - bY)^r \quad &= d^r X^r - r d^{r-1} b X^{r-1} Y + \cdots + (-1)^r b^r Y^r \\ &= \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} b^\nu d^{r-\nu} X^{r-\nu} Y^\nu, \\ Y^s \mapsto (-cX + aY)^s \quad &= (-c)^s X^s + s(-c)^{s-1} a X^{s-1} Y + \cdots + a^s Y^s \\ &= \sum_{\nu=0}^{s} (-1)^{s-\nu} \binom{s}{\nu} a^\nu c^{s-\nu} X^{s-\nu} Y^\nu. \end{aligned}$$

Thus depending on the prime divisors of the binomial coefficients there are some anomalies in prime characteristics.

The Lie algebra $\mathfrak{sl}_2(k)$ consists of the 2×2-matrices with trace 0,

$$\mathfrak{sl}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \,\middle|\, a, b, c \in k \right\}.$$

It acts on the polynomial ring $k[X, Y]$ by derivations, starting with the formulas

$$\text{(2)} \qquad \begin{array}{ll} X & \mapsto -aX - bY, \\ Y & \mapsto -cX + aY, \end{array} \quad \text{for } A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(k).$$

(The easiest way to remember the formulas (2) is by using dual numbers [1, Section 9.5], that is considering $SL_2(k[\delta])$ where $\delta^2 = 0$.) In particular

$$\begin{aligned} X^r &\mapsto r X^{r-1}(-aX - bY), \\ Y^s &\mapsto s\, Y^{s-1}(-cX + aY). \end{aligned}$$

Let $R = k[X, Y]$ be the polynomial ring and R_m be its homogeneous part of degree m, an SL_2-invariant subspace of R with $\dim_k R_m = m + 1$.

Remarks

1. We'll calculate the stabilizers of some homogeneous polynomials in $k[X,Y]$. If H is the stabilizer in $SL_2(K)$ for an extension field $K \supseteq k$, then $H \cap SL_2(k)$ is the stabilizer in $SL_2(k)$. This allows us to freely retreat to the algebraic closure $\bar{k}$ of our base field k.

2. Statements about the dimension or the closure of an orbit refer to the orbit over $\bar{k}$, often without explicit mention. This also applies to other geometric objects.

3. By definition an element A of the Lie algebra "stabilizes" a vector v in an SL_2-module if and only if $Av = 0$. (Think of Av as a displacement.)

4. In characteristic 0 the R_m exactly represent the irreducible SL_2-modules. We won't use this fact.

5. Let $p = \operatorname{char} k$, $m = s\,p^t$ with $s \geq 1$, $t \geq 1$, $p \nmid s$. Let $w \in R_s$ and $v = w^{p^t} \in R_m$. Then the stabilizers in $G = SL_2(k)$ coincide: $G_v = G_w$. This follows from the injectivity of the p^t-th power map:
$$g \cdot v = v \Longleftrightarrow g \cdot v^{p^t} = (g \cdot v)^{p^t} = v^{p^t}.$$

1.2 Some elements and subgroups of SL_2

If primitive n^{th} roots of unity exist in k we distiguish one of them and denote it by ε_n. We consider the matrices

$$\begin{aligned}
\Delta(t) &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{with } t \in k^\times, \\
I &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{with } i^2 = -1, \text{ i.e. } \begin{cases} i = \varepsilon_4 & \text{if } \operatorname{char} k \neq 2, \\ i = 1 & \text{if } \operatorname{char} k = 2, \end{cases} \\
J &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
A(b) &= \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{with } b \in k,
\end{aligned}$$

in $SL_2(k)$. The order of $\Delta(\varepsilon_n)$ is n (if ε_n exists in k). Since $J^2 = I^2 = -\mathbf{1}$, the order of I and of J is 4 if $\operatorname{char} k \neq 2$. If $\operatorname{char} k = 2$, then $J = I$, and its order is 2.

Here are some relevant subgroups and Lie algebras:

$T = \left\{ \Delta(t) |\ t \in k^\times \right\}$, the canonical maximal torus of SL_2,

with corresponding Lie algebra $\mathfrak{t} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \middle|\ a \in k \right\}$,

$N = N_G(T) = T \cup TJ$ the normalizer of T (note $J\Delta(t)J^{-1} = \Delta(t^{-1})$), the canonical Cartan subgroup of SL_2,

$U = \{A(b) \mid b \in k\}$, the canonical maximal unipotent subgroup of SL_2,

with corresponding Lie algebra $\mathfrak{u} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle|\ b \in k \right\}$,

$T_n = \{\Delta(\varepsilon_n)^q \mid 0 \leq q \leq n-1\}$, a cyclic subgroup of $SL_2(\bar{k})$ of order n,

$T_n(k) = \{\Delta(\varepsilon) \mid \varepsilon^n = 1\}$, a cyclic group of order $\#T_n(k) \mid n$,

$N_n(k) = T_n(k) \cup T_n(k)J$, an extension of $T_n(k)$ of order 2 if n is even or if $\operatorname{char} k = 2$, a dihedral group.

1.3 Stability

Let G be an affine algebraic group over an algebraically closed field k. Let V be a finite-dimensional rational G-module. A point $x \in V$ is called

unstable if 0 is in the closure of the orbit $G \cdot x$, i. e. $0 \in \overline{G \cdot x}$,

semistable if x is not unstable, i. e. if $0 \notin \overline{G \cdot x}$,

stable if $x \neq 0$ and the orbit $G \cdot x$ is closed and of maximal dimension (among all orbits),

properly stable if $x \neq 0$, the orbit $G \cdot x$ is closed, and the stabilizer G_x is finite.

As consequences of the Hilbert-Mumford criterion, see [5, Section 2.4.1] or [3] we get:

Proposition 1 *Let $G = SL_2$ and $F \in R_m$ a binary form of degree m. Then*

(i) *F is unstable if and only if F has a linear factor of multiplicity $> \frac{m}{2}$.*

(ii) *F is semistable if and only if all linear factors of F have multiplicity $\leq \frac{m}{2}$.*

(iii) *(For $m \geq 3$) F is stable if and only if F has only linear factors of multiplicity $< \frac{m}{2}$. In this case F is even properly stable.*

Remember that a homogeneous binary form has a decomposition into linear factors, unique up to the sequential order and up to scalar multiples.

1.4 Types of SL_2-orbits

In this section and the next one we again assume that k is an algebraically closed field. Let V be a finite-dimensional rational SL_2-module. For $v \in V$ the orbit $G \cdot v \subseteq V$ has dimension ≤ 3. We also consider the connected component G_v^{o} of the stabilizer G_v in $G = SL_2$.

- If $\dim G \cdot v = 3$, then the stabilizer of v is finite, hence $G_v^{\mathrm{o}} = \mathbf{1}$, the trivial subgroup.
- If $\dim G \cdot v = 2$, then the stabilizer has dimension 1, hence G_v^{o} is conjugated with one of the subgroups T or U, see [1, Chap. 20].
- The assumption $\dim G \cdot v = 1$ leads to a contradiction: It implies that $\dim G_v^{\mathrm{o}} = 2$. But the only 2-dimensional subgroups of SL_2 are the Borel subgroups, hence, see [1, Sect. 4.6 and Exercise 4 of Chap. 12], the orbit would be homeomorphic with the projective variety G/B where $B = TU$ is the canonical Borel subgroup, hence be a complete variety. However an affine variety has no complete subvarieties.
- If $\dim G \cdot v = 0$, then the stabilizer is G itself, and v is a fixed point for the action of G.

This enumeration suggests a taxonomy of SL_2-orbits in a rational SL_2-module V. Let $\partial(G \cdot v) = \overline{G \cdot v} - G \cdot v$ denote the border of the orbit.

(I) $\dim G \cdot v = 3$, $G_v^{\mathrm{o}} = \mathbf{1}$. Then $\partial(G \cdot v)$ may contain some orbits of dimension 2—finitely many of them—and fixed points. More precisely we distinguish between five subcases:

   (a) $G \cdot v$ is closed, i. e. $\partial(G \cdot v)$ is empty.

   (b) $\partial(G \cdot v)$ has dimension 0, hence consists of finitely many fixed points.

   (c) $\partial(G \cdot v)$ has dimension 1, hence consists of a one-dimensional algebraic subset of fixed points.

   (d) $\partial(G \cdot v)$ consists of finitely many closed orbits of dimension 2.

   (e) $\partial(G \cdot v)$ contains finitely many 2-dimensional orbits and some fixed points.

(II) $\dim G \cdot v = 2$, $G_v^{\mathrm{o}} \sim T$. Then $\partial(G \cdot v)$ is empty or consists of fixed points. There is a $g \in G$ such that $w = g \cdot v$ is contained in the fixed point set V^T of the maximal torus T. There are three subcases:

   (a) $G \cdot v$ is closed.

   (b) $\partial(G \cdot v)$ has dimension 0, hence consists of finitely many fixed points.

(c) $\partial(G \cdot v)$ has dimension 1, hence consists of a one-dimensional algebraic subset of fixed points.

(III) $\dim G \cdot v = 2$, $G_v^{\circ} \sim U$. Then G_v is not reductive, hence $G \cdot v$ is not closed, see [3], and $\partial(G \cdot v)$ consists of fixed points. There is a $g \in G$ such that $w = g \cdot v$ is contained in the fixed point set V^U of the unipotent subgroup U. There are two variants:

(a) $\partial(G \cdot v)$ has dimension 0, hence consists of finitely many fixed points.

(b) $\partial(G \cdot v)$ has dimension 1, hence consists of a one-dimensional algebraic subset of fixed points.

(IV) v is a fixed point.

For the types Ib, Ic, IIb, IIc, IIIa, and IIIb the border $\partial(G \cdot v)$ consists of fixed points. Thus

$$\partial(G \cdot v) = \overline{G \cdot v} \cap V^G$$

where $V^G \subseteq V$ is the linear subspace of fixed points.

1.5 Fixed binary forms

The classification of orbits in 1.4 suggests that determining the fixed point subspaces of G, T, and U of an SL_2-module might be useful.

We start with the action of the maximal torus $T \leq G$ on the SL_2-module R_m. Its elements $\Delta(t)$ transform $X^r \mapsto t^{-r} X^r$ and $Y^s \mapsto t^s Y^s$. Thus applying $\Delta(t)$ to

$$v = \sum_{\nu=0}^{m} a_\nu X^{m-\nu} Y^\nu$$

yields the result

$$\sum_{\nu=0}^{m} a_\nu t^{2\nu-m} X^{m-\nu} Y^\nu.$$

Hence v is a fixed point of T if and only if

$$a_\nu t^{2\nu-m} = a_\nu \quad \text{for all } t \in k^\times \text{ and all } \nu = 1, \ldots, m.$$

Since k is assumed as algebraically closed, hence infinite, this forces $a_\nu = 0$ except when $t^{2\nu-m} = 1$ constant, i. e. when $m = 2\nu$.

Proposition 2 *The binary forms of degree m that are fixed by T form the subspace*

$$R_m^T = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ kX^rY^r & \text{if } m = 2r \text{ is even.} \end{cases}$$

In a similar way we determine the fixed points of the maximal unipotent subgroup $U \leq G$ on R_m. Its elements $A(-b)$ (minus sign for convenience) transform $X^r \mapsto (X+bY)^r$ and $Y^s \mapsto Y^s$. In particular U fixes Y^m. Applying $A(-b)$ to

$$v = \sum_{\nu=0}^{m} a_\nu X^{m-\nu} Y^\nu$$

yields the result

$$\sum_{\nu=0}^{m} a_\nu (X+bY)^{m-\nu} Y^\nu = a_0 X^m + (a_1 + a_0 mb) X^{m-1} Y + \cdots + (a_m + P(b)) Y^m$$

where $P \in k[Z]$ is the polynomial

$$P = a_0 Z^m + a_1 Z^{m-1} + \cdots + a_{m-1} Z.$$

Hence if v is a fixed point of U then $P(b) = 0$ for all $b \in k$. Since k is assumed as algebraically closed, hence infinite, this forces $P = 0$, or $a_\nu = 0$ for $\nu = 0, \ldots, m-1$.

Proposition 3 *The binary forms of degree m that are fixed by U form the subspace*

$$R_m^U = kY^m.$$

For the fixed points of the whole group $G = SL_2$ we have $R_m^G \subseteq R_m^T \cap R_m^U$. Thus:

Proposition 4 *The binary forms of degree m that are fixed by $G = SL_2$ form the trivial subspace $R_m^G = 0$.*

Since 0 is the only fixed point in R_m the general taxonomy of orbits is somewhat simplified:

Proposition 5 *The SL_2-orbit of every binary form of degree m is of one of the following types* Ia–e, II, III, *or* IV*.*

If m is odd, then the SL_2-orbit of every binary form of degree m is of one of the types Ia–c, Ie, III, *or* IV*.*

(I) $\dim G \cdot v = 3$, $G_v^{\mathrm{o}} = \mathbf{1}$.

(a) $G \cdot v$ is closed, v is (properly) stable, all its linear factors have multiplicities $< m/2$.

(b) $\partial(G \cdot v)$ consists of the unique fixed point 0, v is unstable, hence has a linear factor of multiplicity $> m/2$.

(c) Void.

(d) $\partial(G \cdot v)$ consists of finitely many closed orbits of dimension 2, v is semistable but not stable, all its linear factors have multiplicities $\leq m/2$, and at least one of them has multiplicity $= m/2$. (This case may occur only if m is even.)

(e) $\partial(G \cdot v)$ contains finitely many 2-dimensional orbits and the fixed point 0, thus v is unstable and has a linear factor of multiplicity $> m/2$.

(II) $\dim G \cdot v = 2$, $G_v^{\mathrm{o}} \sim T$. Then $\partial(G \cdot v)$ is empty or consists of the only fixed point 0. If $R_m^T \neq \{0\}$ we conclude that $m = 2r$ is even, and there are $g \in SL_2$ and $c \in k^\times$ such that $g \cdot v = cX^rY^r$. Thus v has two different linear factors of multiplicities $r = m/2$, hence is semistable, hence $0 \notin \overline{G \cdot v}$. We conclude that $G \cdot v$ is closed. (This case may occur only if m is even.)

If $m = 2$, then v is even stable, see Section 2.2. If $m \geq 4$, then v is not stable by Proposition 1 (iii). Or we use the results from below that show the existence of three-dimensional orbits in R_m, for example that of $X^{m-1}Y + XY^{m-1}$, see Section 5.3 for $m = 4$, and 5.5 for $m \geq 6$.

(III) $\dim G \cdot v = 2$, $G_v^{\mathrm{o}} \sim U$, $\partial(G \cdot v)$ consists of the only fixed point 0. Moreover there are $g \in SL_2$ and $c \in k^\times$ such that $g \cdot v \in R_m^U$, or $g \cdot v = cY^m$. In particular v is unstable. Since $\Delta(t) \cdot Y^m = t^mY^m$ the orbit of Y^m contains the entire line $k^\times Y^m$, thus it is the only one of this type in R_m.

(IV) v is the fixed point 0.

Note that we explicitly know all the orbits of types II, III, and IV. For odd m the types Id and II are impossible.

For type II we conclude that v is (a scalar multiple of) a product of two different linear forms taken to the $m/2$-th power. For type III likewise v is (a scalar multiple of) the m^{th} power of a linear form. Since k is algebraically closed we may absorb the scalar factors into the linear forms.

Moreover for type Id we conclude that exactly one of the linear factors has multiplicity $= m/2$.

Corollary 1 *Let $v \in R_m$. Then the following statements are equivalent:*

(i) *The SL_2-orbit of v has dimension* 2*.*

(ii) *The stabilizer of v has dimension* 1*.*

(iii) *v is the m^{th} power of a non-zero linear form, or (only for even $m = 2r$) the r^{th} power of a product of two different non-zero linear forms.*

Corollary 2 *The only SL_2-orbits in R_m of dimension < 3 are those of* 0*, Y^m, and (if $m = 2r$ is even) cX^rY^r with $c \in k^\times$.*

Corollary 3 *Let $v \in R_m$ be a binary form of degree m all of whose linear factors have multiplicities $\leq \frac{m}{2}$, and assume that the stabilizer of v has dimension* 1*. Then $m = 2r$ is even and the SL_2-orbit of v is closed and meets cX^rY^r.*

Proof. The orbit has dimension 2. The assumption on the multiplicities rules out type III. Thus we have type II. $\diamond$

Call two linear factors essentially different if they are not scalar multiples of each other.

Corollary 4 *Assume that $v \in R_m$ has at least three essentially different linear factors. Then the stabilizer G_v is finite.*

Proof. By Corollary 2 binary forms with orbits of dimension < 3 have at most two essentially different linear factors. $\diamond$

Corollary 5 *Assume that $v \in R_m$ has a linear factor of multiplicity $\neq m$ or $\frac{m}{2}$. Then the stabilizer G_v is finite.*

Résumé: Except in the very few cases listed in Corollary 2 the stabilizer of a binary form is finite.

2 Low Dimensions

We start the study of the stabilizers and orbits of sample concrete binary forms with the low degrees up to 3.

2.1 The case $m = 1$ of linear forms

Let us start with the trivial case $m = 1$, $\dim R_m = 2$, and consider the element $v = Y \in R_1$. Setting $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k)$ we see that the orbit contains $g \cdot v = -cX + aY$, that is an arbitrary element $\neq 0$ of R_1. Thus $G = SL_2(k)$ has (besides $G \cdot 0 = 0$) exactly one orbit, $G \cdot Y = R_1 - \{0\}$, that is obviously not closed in R_1. Since $g \cdot v = v$ if and only if $a = 1$ and $c = 0$, the stabilizer G_v is the maximal unipotent subgroup U. A Lie algebra element $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g} = \mathfrak{sl}_2(k)$ annihilates v if and only if $c = a = 0$. Thus the stabilizer in the Lie algebra is $\mathfrak{g}_v = \mathfrak{u} = \mathrm{Lie}(G_v)$. In particular the orbit map is separable. The result is:

Proposition 6 *The stabilizer of the homogeneous polynomial Y in the group $G = SL_2(k)$ is the maximal unipotent subgroup U. The orbit of Y contains 0 in its closure hence is not closed, $\partial(G \cdot Y) = \{0\}$, and Y is unstable. The orbit map is separable.*

(For the second sentence we assume that k is algebraically closed.)

Corollary 1 *Let $Z \in R_1$ be a nonzero linear form. Then the stabilizer of Z in the group $SL_2(k)$ a maximal unipotent subgroup. The orbit of Z contains 0 in its closure, hence it not closed, $\partial(G \cdot Z) = \{0\}$, and Z is unstable. The orbit map is separable.*

This result has an application to powers of the characteristic:

Corollary 2 *Assume $\mathrm{char}\, k = p > 0$ and $q = p^t$ with $t \geq 1$. Let $Z \in R_1$ be a nonzero linear form. Then the stabilizer of the homogeneous polynomial $Z^q \in R_q$ in the group $SL_2(k)$ is the same maximal unipotent subgroup as the stabilizer of Z. The orbit contains $0 \in R_q$ in its closure, hence it not closed but unstable, and $\partial(G \cdot Z^q) = \{0\}$. The stabilizer in the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(k)$ is $\mathfrak{g}$, in particular the orbit map is inseparable.*

Proof. The first statement follows from the injectivity of the q-th power map, see remark 5 in 1.1. The last statement follows since every derivation A annihilates every q-th power: $A(Z^q) = q\, Z^{q-1} A(Z) = 0$. $\diamond$

2.2 The case $m = 2$ of quadratic forms

We won't work out the complete orbit classification for R_2 over arbitrary fields but concentrate on two representative elements, the quadratic forms

1. $v = XY$, an example of a non-degenerate form, of orbit type II,

2. $v = Y^2$, the degenerate case, of orbit type III.

Note that these two quadratic forms in a certain sense are typical representatives: Over the algebraic closure $\bar{k}$ every non-zero quadratic form is either a product of two different linear forms or the square of a linear form. In the first case, the non-degenerate case, as in 2.1 a suitable matrix $\in SL_2(\bar{k})$ transforms the second factor to Y, and then a matrix from the stabilizer of Y transforms the first factor to sX with $s \in \bar{k}^\times$. (In particular under $GL_2(\bar{k})$ the non-degenerate quadratic forms form a single orbit.) Thus up to a scalar multiple (that doesn't change the stabilizer nor the geometric properties of the orbit) we are in case 1, at least over an algebraically closed field. For an instance what might happen over a field that is not algebraically closed see Proposition 16.

In the second case, the square of a linear form, we again transform this linear form to Y, the resulting quadratic form being Y^2.

The non-degenerate case

The matrix $g \in SL_2$ given by (1) maps the form $v = XY$ to

$$(dX - bY)(aY - cX) = -cdX^2 + (ad + bc)XY - abY^2.$$

Hence g stabilizes v if and only if

$$ad - bc = 1, \quad cd = 0, \quad ad + bc = 1, \quad ab = 0.$$

If $c = 0$, then these conditions yield $ad = 1$, $d = 1/a$, $b = 0$, hence $g = \Delta(a) \in T$.

If $c \neq 0$ and $\operatorname{char} k \neq 2$, the conditions yield $d = 0$, $bc = -1$, as well as $bc = 1$, a contradiction. If however $\operatorname{char} k = 2$, we get $c = 1/b$ and $a = 0$, forcing $g = \Delta(b)\, I$, a matrix $\in N$ that stabilizes v.

The Lie algebra element $A \in \mathfrak{sl}_2(k)$, acting as in (2), maps $v = XY$ to

$$X\,(-cX + aY) + (-aX - bY)\,Y = -cX^2 - bY^2.$$

Hence A annihilates XY if and only if $b = c = 0$, that is if and only if $A \in \mathfrak{t}$, the Lie algebra of the torus T. In particular the orbit map is separable.

Proposition 7 *The stabilizer of the homogeneous polynomial XY in the group $SL_2(k)$ is the maximal torus T if* $\operatorname{char} k \neq 2$*, the Cartan subgroup N if* $\operatorname{char} k = 2$*. The orbit of XY is closed. The orbit map is separable.*

Corollary 1 *The quadratic form XY, or more generally every non-degenerate quadratic form, is stable for SL_2.*

The degenerate case

The matrix $g \in G = SL_2$ given by (1) maps the form $v = Y^2$ to

$$(aY - cX)^2 = c^2X^2 - 2acXY + a^2Y^2.$$

Hence g stabilizes v if and only if

$$ad - bc = 1, \quad c^2 = 0, \quad 2ac = 0, \quad a^2 = 1,$$

that is if and only if $c = 0$, $a = \pm 1$, $d = 1/a$. Therefore $G_v = U \cup -U$ if $\operatorname{char} k \neq 2$, $G_v = U$ if $\operatorname{char} k = 2$.

The Lie algebra element $A \in \mathfrak{sl}_2(k)$, acting as in (2), maps $v = Y^2$ to

$$2Y\,(-cX + aY) = -2cXY + 2aY^2.$$

In characteristic 2 every $A \in \mathfrak{sl}_2(k)$ annihilates Y^2, hence the orbit map is inseparable. In characteristic $\neq 2$ the matrix A annihilates Y^2 if and only if $a = c = 0$, that is if and only if $A \in \mathfrak{u}$, the Lie algebra of the unipotent group U. In particular the orbit map is separable.

Proposition 8 *The stabilizer of the homogeneous polynomial Y^2 in the group $G = SL_2(k)$ is the maximal unipotent subgroup U if $\operatorname{char} k = 2$, the extension $U \cup -U$ of order 2 if $\operatorname{char} k \neq 2$. The orbit of Y^2 is not closed but $\partial(G \cdot Y^2) = \{0\}$. The orbit map is separable if and only if $\operatorname{char} k \neq 2$.*

Corollary 1 *The quadratic form Y^2, or more generally every degenerate quadratic form, is unstable for SL_2.*

2.3 The case $m = 3$ of cubic forms

Here again we'll concentrate on three typical forms:

1. $X^2Y + XY^2 = XY\,(X + Y)$, a product of three different linear factors, of orbit type Ia,
2. XY^2, a product of a square and a different linear form, of orbit type Ib or Ie (it will turn out to be Ie),
3. Y^3, a cube of a linear form, of orbit type III.

Note that every cube of a non-zero linear form may be transformed to Y^3 by $SL_2(k)$, thus the non-zero cubes constitute a single $SL_2(k)$-orbit, and Y^3 is representative for it. In case 2 we may transform the squared linear form to Y^2, and then the different linear form to a scalar multiple of X by $SL_2(k)$ (or to X by $GL_2(k)$). Thus XY^2 is a good representative. The situation with three different linear forms is more complex, we only treat the sample form given in 1. Sections 4.3 and **??** treat the alternative example $X^3 + Y^3$ that has three different linear factors if and only if $\operatorname{char} k \neq 3$.

An example of a product of three different linear forms

The matrix g from (1) transforms $v = X^2Y + XY^2$ to

$$(dX - bY)^2(-cX + aY) + (dX - bY)(-cX + aY)^2 =$$

$$\begin{aligned}
-cd^2X^3 &+ 2bcdX^2Y - b^2cXY^2 \\
&\quad + ad^2X^2Y - 2abdXY^2 + ab^2Y^3 \\
+c^2dX^3 &- 2acdX^2Y + a^2dXY^2 \\
&\quad - bc^2X^2Y + 2abcXY^2 - a^2bY^3.
\end{aligned}$$

The conditions that g is in $SL_2(k)$ and fixes v are equivalent with the equations

$$ad - bc = 1 \tag{3}$$

$$c^2d - cd^2 = 0 \tag{4}$$

$$ad^2 + 2bcd - 2acd - bc^2 = 1 \tag{5}$$

$$a^2d + 2abc - b^2c - 2abd = 1 \tag{6}$$

$$ab^2 - a^2b = 0 \tag{7}$$

Equation (4) is equivalent with $cd(c - d) = 0$, Equation (7) with $ab(b - a) = 0$. Therefore we distinguish three cases: $c = 0$ or $d = 0$ or $c = d$, and in the third case we distinguish between $a = 0$ or $b = 0$ or $a = b$.

Case 1, $c = 0$. We get (3) $ad = 1$ and (5) $ad^2 = 1$, hence $a = d = 1$, and (6) $1 - 2b = 1$. For $\operatorname{char} k \neq 2$ we conclude $b = 0$, therefore $g = \mathbf{1}$ is the unit matrix. For $\operatorname{char} k = 2$ we get the solution $g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = A(b)$ where $b \in k$ is restricted by (7) $b^2 - b = 0$, hence $b = 0$ or $b = 1$.

Case 2, $d = 0$, $c \neq 0$. We get (3) $-bc = 1$ and (5) $-bc^2 = 1$, hence $c = 1$, $b = -1$, and (6) $-2a - 1 = 1$. For $\operatorname{char} k \neq 2$ we conclude $a = -1$, therefore g is the matrix

$$B = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

a matrix of multiplicative order 3. For $\operatorname{char} k = 2$ we get the solution $g = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ where $a \in k$ is restricted by (7) $a + a^2 = 0$, hence $a = 0$ or $a = 1$.

Case 3, $c = d$ with $c \neq 0$, $d \neq 0$. If $a = 0$ the conditions boil down to $-bc = 1$ and $-bc^2 = 1$, yielding $b = 1$ and $c = -1$, and g is the matrix

$$B^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

If $a \neq 0$, but $b = 0$ we get the conditions $ad = 1$, $-ad^2 = ad^2 - 2acd = 1$, $a^2d = 1$ that are contradictory if $\operatorname{char} k \neq 2$. If $\operatorname{char} k = 2$ the conditions

$ad = 1$, $ad^2 = 1$, $a^2d = 1$, have the solution $a = d = c = 1$, yielding $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. In the third subcase, $a \neq 0$, $b \neq 0$, $b = a$ equation (3) yields the contradiction $ad - ad = 1$ (in any characteristic).

This is the result:

Proposition 9 *The stabilizer of the homogeneous polynomial $X^2Y + XY^2$ in the group $SL_2(k)$ is finite, more exactly it is*

(i) *the cyclic subgroup of order 3 generated by the matrix B and consisting of*

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

if $\operatorname{char} k \neq 2$.

(ii) *the group of order 6 generated by J and $A(1)$, consisting of*

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and isomorphic with the symmetric group $\mathcal{S}_3$ if $\operatorname{char} k = 2$.

The Lie algebra action of $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(k)$ as in (2) maps $X^2Y + XY^2$ to

$$-cX^3 - (a + 2c)X^2Y + (a - 2b)XY^2 - bY^3.$$

Hence A annihilates $X^2Y + XY^2$ if and only if $c = 0$, $b = 0$, and $a = 0$, regardless of the characteristic.

Corollary 1 *The stabilizer of the homogeneous polynomial $X^2Y + XY^2$ in the Lie algebra $\mathfrak{sl}_2(k)$ is 0. In particular the orbit map is separable.*

A cubic with a square factor

To decide between the orbit types Ib and Ie we consider the matrix $\begin{pmatrix} a & -1/a^2 \\ 0 & 1/a \end{pmatrix} \in SL_2(k)$. It transforms

$$XY^2 \mapsto aXY^2 + Y^3.$$

The specialization $a \to 0$ shows that $Y^3 \in \partial(G \cdot v)$. Therefore the border doesn't consist of 0 alone, and the orbit is of type Ie.

The cubic form $v = XY^2$ is mapped to

$$(dX - bY)(-cX + aY)^2 = c^2dX^3 - (2acd + bc^2)X^2Y + (a^2d + 2abc)XY^2 - a^2bY^3$$

by $g \in SL_2(k)$. Therefore g stabilizes v if and only if

$$ad - bc = 1, \quad c^2d = 0, \quad a^2b = 0, \quad 2acd + bc^2 = 0, \quad 2abc + a^2d = 1.$$

Assuming $b \neq 0$ we get $a = 0$, $-bc = 1$, $bc^2 = 0$, a contradiction.

Hence $b = 0$, $ad = 1$, $a^2d = 1$, $a = 1$, $d = 1$, $c = 0$. Thus the stabilizer consists of the unit matrix only.

The Lie algebra action of $A \in \mathfrak{sl}_2(k)$ maps v to

$$(-aX - bY)\, Y^2 + 2XY\,(-cX + aY) = -2cX^2Y + aXY^2 - bY^3.$$

This is 0 if and only if $a = b = 0$, $2c = 0$. These conditions force $A = 0$ if $\operatorname{char} k \neq 2$. If $\operatorname{char} k = 2$ they force $A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ with arbitrary $c \in k$.

Proposition 10 *Let $v \in R_3$ be the homogeneous polynomial XY^2.*

(i) *The stabilizer of v in the group $SL_2(k)$ is the trivial subgroup* $\mathbf{1}$.

(ii) *v is unstable, the orbit of v is not closed, its closure contains* 0 *as well as the unique two-dimensional orbit $G \cdot Y^3$.*

(iii) *If* $\operatorname{char} k \neq 2$, *then the stabilizer of v in the Lie algebra $\mathfrak{sl}_2(k)$ is* 0. *The orbit map is separable.*

(iv) *If* $\operatorname{char} k = 2$, *then the stabilizer of v in the Lie algebra $\mathfrak{sl}_2(k)$ is nilpotent of dimension* 1. *The orbit map is inseparable.*

A cube

The cubic form $v = Y^3$ is mapped to

$$(-cX + aY)^3 = -c^3X^3 + 3ac^2X^2Y - 3a^2cXY^2 + a^3Y^3$$

by $g \in SL_2(k)$. Therefore g stabilizes v if and only if $c = 0$ and $a^3 = 1$, or if and only if

$$g = \begin{pmatrix} \varepsilon & b \\ 0 & 1/\varepsilon \end{pmatrix} \tag{8}$$

where ε is a 3^{rd} root of unity and $b \in k$ arbitrary.

The Lie algebra action of $A \in \mathfrak{sl}_2(k)$ maps v to

$$3Y^2\,(-cX + aY) = -3cXY^2 + 3aY^3.$$

This is 0 if and only if $3a = 3c = 0$. We summarize:

Proposition 11 *Let $v \in R_3$ be the homogeneous polynomial Y^3.*

(i) *The stabilizer G_v of v in the group $G = SL_2(k)$ is the subgroup consisting of the elements* (8)*, a finite extension of the maximal unipotent subgroup U.*

(ii) *The orbit of v is not closed, and $\partial(G \cdot v) = \{0\}$.*

(iii) *If* $\operatorname{char} k \neq 3$*, then the stabilizer of v in the Lie algebra $\mathfrak{sl}_2(k)$ is* $\mathfrak{u} = \operatorname{Lie}(U) = \operatorname{Lie}(G_v)$*. The orbit map is separable.*

(iv) *If* $\operatorname{char} k = 3$*, then the stabilizer of v in the Lie algebra $\mathfrak{sl}_2(k)$ is $\mathfrak{sl}_2(k)$. The orbit map is inseparable.*

In characteristic 3 this result repeats Corollary 2 in 2.1.

3 Stabilizer and Orbit of Y^m and X^rY^r

For general m we consider some selected orbits only, starting with the "exceptional" orbit types II and III.

3.1 The binary form Y^m

The last example easily extends from Y^3 to $v = Y^m$. The matrix g from (1) transforms v to

$$(-cX + aY)^m = (-c)^m X^m + \cdots + a^m Y^m.$$

Hence g stabilizes v if and only if $c = 0$ and $a^m = 1$, that is if and only if

$$g = \begin{pmatrix} \varepsilon & b \\ 0 & 1/\varepsilon \end{pmatrix} \tag{9}$$

where ε is an m^{th} root of unity and $b \in k$ arbitrary.

The derivation $A \in \mathfrak{sl}_2(k)$ as in (2) maps v to

$$mY^{m-1}(-cX + aY) = -mcXY^{m-1} + maY^m.$$

hence annihilates v if and only if $ma = mc = 0$, that is in any case if $\operatorname{char} k \mid m$, and if and only if $a = c = 0$ if $\operatorname{char} k \nmid m$.

We summarize:

Proposition 12 *Let $v \in R_m$ be the homogeneous polynomial Y^m.*

(i) *The stabilizer G_v of v in the group $G = SL_2(k)$ is the subgroup consisting of the elements* (9)*, a finite extension of the maximal unipotent subgroup U.*

(ii) *The orbit of v is not closed, and $\partial(G \cdot v) = \{0\}$.*

(iii) *If $\operatorname{char} k \nmid m$, then the stabilizer of v in the Lie algebra $\mathfrak{sl}_2(k)$ is $\mathfrak{u} = \mathrm{Lie}(U) = \mathrm{Lie}(G_v)$. The orbit map is separable.*

(iv) *If $\operatorname{char} k \mid m$, then the stabilizer of v in the Lie algebra $\mathfrak{sl}_2(k)$ is $\mathfrak{sl}_2(k)$. The orbit map is inseparable.*

3.2 The binary form X^rY^r

The orbit is of type II, in particular it is closed. Since the diagonal matrix $\Delta(t)$, $t \in k^\times$, maps $X^r \mapsto t^{-r}X^r$ and $Y^r \mapsto t^rY^r$, T stabilizes X^rY^r, and T has finite index in the stabilizer.

Moreover J maps $X \mapsto -Y$ and $Y \mapsto X$. Hence for an even r (or in characteristic 2 for any r) also J stabilizes X^rY^r, and so does the whole Cartan subgroup N.

The case $r = 1$ was treated in 2.2.

The Lie algebra action

The matrix $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g} = \mathfrak{sl}_2(k)$ maps $v = X^rY^r$ to

$$rX^{r-1}Y^r\,(-aX-bY)+rX^rY^{r-1}\,(-cX+aY) = -rbX^{r-1}Y^{r+1}-rcX^{r+1}Y^{r-1}.$$

This is 0 if and only if $\operatorname{char} k \mid r$ or $b = c = 0$. Thus

$$\mathfrak{g}_v = \begin{cases} \mathfrak{t} & \text{if } \operatorname{char} k \nmid r, \\ \mathfrak{g} & \text{if } \operatorname{char} k \mid r. \end{cases}$$

Since v is a pure r^{th} power, the orbit map is separable if and only if $\operatorname{char} k \nmid r$.

The case $r \geq 2$ even

The matrix g from (1) transforms $v = X^rY^r$ to

$$(dX - bY)^r(-cX + aY)^r = c^rd^rX^{2r} - \ldots + a^rb^rY^{2r}.$$

If g stabilizes v, then $ab = 0$, hence $a = 0$ or $b = 0$, and $cd = 0$. If $b = 0$, then $ad = 1$, $d = 1/a \neq 0$, $c = 0$, thus $g \in T$. If $a = 0$, then $-bc = 1$, $c = -1/b \neq 0$, $d = 0$, thus $g \in TJ$.

In summary we have shown:

Proposition 13 *Let $r \geq 2$ be even. Then the stabilizer of the homogeneous polynomial X^rY^r in the group $SL_2(k)$ is the Cartan subgroup N. The orbit is closed, and the orbit map is*

(i) *separable if* $\operatorname{char} k \nmid r$,

(ii) *inseparable if* $\operatorname{char} k \mid r$.

The case $r \geq 1$ odd

The matrix g from (1) transforms $v = X^rY^r$ to

$$-c^rd^rX^{2r} + \ldots - a^rb^rY^{2r}.$$

If g stabilizes v, then $ab = 0$, hence $a = 0$ or $b = 0$, and $cd = 0$. If $b = 0$, then $ad = 1$, $d = 1/a \neq 0$, $c = 0$, thus $g \in T$. If $a = 0$, then $-bc = 1$, $c = -1/b \neq 0$, $d = 0$, thus $g \in TJ$. However J maps X^rY^r to $-X^rY^r$, contradiction for $\operatorname{char} k \neq 2$.

In summary we have shown:

Proposition 14 *Let $r \geq 1$ be odd. Then the stabilizer of the homogeneous polynomial X^rY^r in the group $SL_2(k)$ is*

(i) *the maximal torus T if* $\operatorname{char} k \neq 2$,

(ii) *the Cartan subgroup N if* $\operatorname{char} k = 2$.

The orbit is closed, and the orbit map is

(iii) *separable if* $\operatorname{char} k \nmid r$,

(iv) *inseparable if* $\operatorname{char} k \mid r$.

For $r = 1$ this result repeats Proposition 7.

Résumé

(For (ii) and (iii) we assume k to be algebraically closed.)

Theorem 1 *Let $r \geq 1$ and v be the binary form $v = X^rY^r$ of degree $2r$. Then:*

(i) *The stabilizer of v in $G = SL_2(k)$ is*

- *the canonical maximal torus T if r is odd and* $\operatorname{char} k \neq 2$,
- *the Cartan subgroup $N = N_G(T)$ if r is even or if* $\operatorname{char} k = 2$.

(ii) *The G-orbit of v is closed, and v is*

- *stable if $r = 1$,*
- *semistable but not stable if $r \geq 2$.*

(iii) *The orbit map $G \longrightarrow G \cdot v$ is separable if and only if* $\operatorname{char} k \nmid r$.

4 Stabilizer and Orbit of $X^r + Y^r$

Here is the result for an algebraically closed field k. The proofs of its single parts are in sections 4.1 to 4.6, as well as the explicit determination of the stabilizer (in most cases) over an arbitrary field.

Theorem 2 *Assume that k is algebraically closed. Let $r \geq 1$ and v be the binary form $v = X^r + Y^r$ of degree r. Then:*

(i) *The stabilizer of v in $G = SL_2(k)$ is*

- *a maximal unipotent subgroup if r is a power of* $\operatorname{char} k$ *(including $r = 1$),*
- *a maximal torus if r is twice a power of* $\operatorname{char} k$ *(including $r = 2$) and* $\operatorname{char} k \neq 2$,
- *finite otherwise.*

(ii) *The G-orbit of v*

- *is not closed but $\partial(G \cdot v) = \{0\}$, if r is a power of* $\operatorname{char} k$,
- *is closed otherwise,*

and v is

- *unstable if r is a power of* $\operatorname{char} k$,
- *semistable but not stable if* $\operatorname{char} k \neq 2$, *$r \geq 3$, and r is twice a power of* $\operatorname{char} k$,
- *stable otherwise.*

(iii) *The orbit map $G \longrightarrow G \cdot v$ is separable if and only if* $\operatorname{char} k \nmid r$.

The proof is in 4.1 for $r = 1$ and $r = 2$, in 4.2 for $r \geq 3$.

A general remark: The diagonal matrix $\Delta(t)$ maps X^r to $t^{-r}X^r$ and Y^r to $t^r Y^r$. Hence for an r^{th} root of unity $\varepsilon \in k$ the diagonal matrix $\Delta(\varepsilon)$ fixes $v = X^r + Y^r$. Thus the stabilizer G_v contains the cyclic subgroup $T_r(k)$ consisting of the r^{th} roots of unity in k. If k contains a primitive r^{th} root of unity, then $T_r(k) = T_r$ has order r.

4.1 The cases $r = 1$ and $r = 2$

The case $r = 1$ was implicitly treated in Section 2.1. Instead of determining G_v directly we note that $h = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ maps $u = Y$ to $v = X + Y$, thus $G_v = hG_uh^{-1}$. The orbit is of type III.

Proposition 15 *The stabilizer of the homogeneous polynomial $X+Y$ in the group $G = SL_2(k)$ is the maximal unipotent subgroup hUh^{-1} and consists of the matrices $\begin{pmatrix} 1+b & b \\ -b & 1-b \end{pmatrix}$ for $b \in k$. The binary linear form $v = X+Y$ is unstable, its orbit is not closed but $\partial(G \cdot v) = \{0\}$. The stabilizer in the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(k)$ is the nilpotent subalgebra consisting of the matrices $\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$ for $a \in k$. In particular $\mathfrak{g}_v$ is the Lie algebra of G_v, thus the orbit map is separable.*

Corollary 2 in 2.1 also settles the case of $X^q + Y^q = (X+Y)^q$ for q a power of the characteristic, in particular the case $r = 2$ in characteristic 2.

The case $r = 2$ for $\operatorname{char} k \neq 2$ is implicit in 2.2. Let us calculate the stabilizer explicitly. The matrix g from (1) transforms $v = X^2 + Y^2$ to $(dX - bY)^2 + (aY - cX)^2 =$

$$d^2X^2 - 2bdXY + b^2Y^2 + c^2X^2 - 2acXY + a^2Y^2.$$

The conditions that g is in $SL_2(k)$ and fixes v yield the equations

$$ad - bc = 1, \tag{10}$$
$$c^2 + d^2 = 1, \tag{11}$$
$$ac + bd = 0, \tag{12}$$
$$a^2 + b^2 = 1. \tag{13}$$

First assume $a \neq 0$. Then multiplying (12) by a we get

$$0 = a^2c + abd = (1 - b^2) \cdot c + b \cdot (1 + bc) = c - b^2c + b + b^2c = b + c,$$

hence $c = -b$. If we also assume $b \neq 0$, multiplying (12) by b we likewise get

$$0 = abc + b^2d = a \cdot (-1 + ad) + (1 - a^2) \cdot d = -a + a^2d + d - a^2d = d - a,$$

hence $d = a$. We are left with the condition $a^2 + b^2 = 1$. This defines a maximal torus of G (that over $\bar{k}$ must be conjugated with T).

Proposition 16 *The stabilizer of the homogeneous polynomial $X^2 + Y^2$ in the group $SL_2(k)$*

(i) *(for* $\operatorname{char} k \neq 2$*) is the torus*

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a, b \in k,\ a^2 + b^2 = 1 \right\},$$

and the orbit is closed, the orbit map is separable,

(ii) *(for* $\operatorname{char} k = 2$*) is the maximal unipotent subgroup*

$$\left\{ \begin{pmatrix} 1+b & b \\ -b & 1-b \end{pmatrix} \,\middle|\, b \in k \right\},$$

and $\partial(G \cdot v) = \{0\}$*. The orbit map is inseparable.*

In characteristic 2 the orbit is of type III, otherwise of type II.

4.2 The Lie algebra action and stability for $r \geq 3$

Proposition 17 *Assume* $r \geq 3$*. Then the stabilizer of the homogeneous polynomial* $v = X^r + Y^r$ *in the Lie algebra* $\mathfrak{g} = \mathfrak{sl}_2(k)$ *is*

(i) $\mathfrak{g}_v = \mathfrak{g}$ *if* $\operatorname{char} k \,|\, r$*. In particular the orbit map for* v *under* SL_2 *is inseparable.*

(ii) $\mathfrak{g}_v = 0$ *if* $\operatorname{char} k \nmid r$*. In particular the stabilizer* G_v *in* $G = SL_2(k)$ *is finite, the* SL_2*-orbit of* v *over the algebraic closure* $\bar{k}$ *has dimension* 3*, and the orbit map is separable.*

Proof. The Lie algebra element $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g} = \mathfrak{sl}_2(k)$ transforms $v = X^r + Y^r$ to

$$r\,X^{r-1}AX + r\,Y^{r-1}AY = -raX^r - rb\,X^{r-1}Y - rc\,XY^{r-1} + raY^r,$$

For (i) we immediately conclude that $Av = 0$.

For (ii) we see that A annihilates v if and only if $a = b = c = 0$, that is $A = 0$. The additional statements are immediate consequences. $\diamond$

Proposition 18 *Let* $r \geq 3$*. Then the* SL_2*-orbit of* $v = X^r + Y^r$ *is*

(i) *closed (over* $\bar{k}$*) and properly stable if* $\operatorname{char} k \nmid r$ *(type I),*

(ii) *not closed with* $\partial(G \cdot v) = \{0\}$*, and unstable if* r *is a power of* $\operatorname{char} k$ *(type III); the stabilizer* G_v *is the maximal unipotent subgroup from Proposition 15,*

(iii) *closed and semistable if* r *is twice a power of* $\operatorname{char} k$ *and* $\operatorname{char} k \neq 2$ *(type II); the stabilizer* G_v *is the torus from Proposition 16,*

(iv) *closed and properly stable if* $\operatorname{char} k \,|\, r$*, but* r *is not a power nor twice a power of* $\operatorname{char} k$ *(type I).*

Proof. In the case of (i) $X^r + Y^r$ decomposes into r different simple linear factors over $\bar{k}$. Thus it is stable and has a closed orbit.

Now assume $p = \operatorname{char} k \mid r$, say $r = sp^t$ with $t > 0$ and $p \nmid s$. We have

$$X^r + Y^r = (X^s + Y^s)^{p^t}$$

where $X^s + Y^s$ decomposes into s different linear factors. Hence $X^r + Y^r$ decomposes into s different linear factors, each of multiplicity p^t. In the case of (ii) we have $s = 1$, thus all linear factors are identical. Hence $X^r + Y^r$ is unstable for the action of SL_2, and its stabilizer equals G_{X+Y}.

In the case of (iii) the multiplicity of the linear factors equals $r/2$, implying semistability. The stabilizer is equal to $G_{X^2+Y^2}$ hence conjugate with T by Proposition 16. The orbit is of type II.

In the case of (iv) the multiplicity is $< \frac{r}{2}$, implying proper stability. $\diamond$

Corollary 1 *Let* $p = \operatorname{char} k$, $r = s\,p^t$ *with* $p \nmid s$. *Let* $v = X^r + Y^r$ *and* $w = X^s + Y^s$. *Then* $G_v = G_w$.

Proof. This is a special case of Remark 5 in 1.1. $\diamond$

This corollary reduces the case $p \mid r$ to the case $p \nmid s$.

4.3 The stabilizer for r odd, $r \geq 3$

By the corollary in 4.2 we may assume that $\operatorname{char} k \nmid r$.

The matrix g from (1) transforms $v = X^r + Y^r$ to $(dX - bY)^r + (aY - cX)^r =$

$$d^r X^r - r d^{r-1} b X^{r-1} Y + \binom{r}{2} d^{r-2} b^2 X^{r-2} Y^2 - \cdots \quad + r d b^{r-1} X Y^{r-1} - b^r Y^r$$
$$- c^r X^r + r c^{r-1} a X^{r-1} Y - \binom{r}{2} c^{r-2} a^2 X^{r-2} Y^2 + \cdots \quad - r c a^{r-1} X Y^{r-1} + a^r Y^r.$$

The conditions that g is in $SL_2(k)$ and fixes v yield the equations

$$ad - bc = 1, \tag{14}$$
$$d^r - c^r = 1, \tag{15}$$
$$a^r - b^r = 1, \tag{16}$$
$$ac^{r-1} - bd^{r-1} = 0, \tag{17}$$
$$a^{r-1}c - b^{r-1}d = 0. \tag{18}$$

First assuming that $\operatorname{char} k \nmid r - 1$ we have $\binom{r}{2} \neq 0$ and get the additional necessary condition

$$a^2 c^{r-2} - b^2 d^{r-2} = 0. \tag{19}$$

Multiplying Equation (17) by a we get

$$0 = a^2c^{r-1} - abd^{r-1} = b^2d^{r-2}c - abd^{r-1} = bd^{r-2} \cdot (bc - ad)$$

using Equation (19). Since the determinant is $ad - bc = 1$, and $r \geq 3$, we conclude that

$$b = 0 \quad \text{or} \quad d = 0.$$

First assume $b \neq 0$. Then necessarily $d = 0$, and our equations collapse to

$$-bc = 1, \quad -c^r = 1, \quad a^r - b^r = 1, \quad ac^{r-1} = 0, \quad \ldots$$

Since $c \neq 0$ by the first one, the fourth one gives $a = 0$. Then from the third one $b^r = -1$, hence $b^rc^r = (-1)(-1) = 1$, contradicting the first equation since r is odd and the characteristic of k is not 2 (note that $r - 1$ is even).

Thus we necessarily have $b = 0$. In this case our equations collapse to

$$ad = 1, \quad d^r - c^r = 1, \quad a^r = 1, \quad ac^{r-1} = 0, \quad \ldots$$

Since $a \neq 0$ by the first one, the fourth one gives $c = 0$. By the third one a is an r^{th} root of unity and d is its inverse.

Since these matrices indeed fix v we have shown:

Proposition 19 *Let $r \geq 3$ be odd. Assume that the characteristic of k doesn't divide $r(r-1)$. Then the stabilizer of the homogeneous polynomial $X^r + Y^r$ in the group $SL_2(k)$ is the cyclic subgroup $T_r(k)$ of order $\mid r$.*

The remaining case where $\operatorname{char} k \mid r - 1$ is considerably more complex. Equation (19) doesn't necessarily hold. Instead we use Equation (18), and multiplying Equation (17) by a^{r-2} we get

$$0 = a^{r-1}c^{r-1} - a^{r-2}bd^{r-1} = b^{r-1}c^{r-2}d - a^{r-2}bd^{r-1} = bd\,(b^{r-2}c^{r-2} - a^{r-2}d^{r-2}).$$

One of the factors must vanish: $b = 0$ or $d = 0$ or $b^{r-2}c^{r-2} - a^{r-2}d^{r-2} = 0$.

The case $b = 0$ results in $ad = 1$, $a^r = 1$, $a^{r-1}c = 0$, $c = 0$,

$$g = \Delta(\varepsilon) \quad \text{with } \varepsilon^r = 1.$$

The case $d = 0$ results in $bc = -1$, $c^r = -1$, $b^r = -1/c^r = 1$ (since r is odd), $ac^{r-1} = 0$, $a = 0$, $b^r = a^r - 1 = -1$, contradiction for $\operatorname{char} k \neq 2$.

If however $\operatorname{char} k = 2$ we get the additional solutions

$$g = \begin{pmatrix} 0 & \varepsilon \\ 1/\varepsilon & 0 \end{pmatrix} = \Delta(\varepsilon)J \quad \text{with } \varepsilon^r = 1.$$

The last (and most complex) subcase to consider is $b \neq 0$ and $d \neq 0$. Then also $a \neq 0$ and $c \neq 0$, but

$$b^{r-2}c^{r-2} - a^{r-2}d^{r-2} = 0.$$

We keep this intermediate result:

Lemma 1 *Let $r \geq 3$ be odd,* $\operatorname{char} k \mid r-1$*, and let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k)$ *stabilize $X^r + Y^r$. Then either $g \in T_r(k)$, or (only if* $\operatorname{char} k = 2$*) $g \in T_r(k)J$, or $abcd \neq 0$ and $(ad)^{r-2} = (bc)^{r-2}$.*

Now in $\bar{k}$ we have the decomposition

$$\prod_{\nu=0}^{r-3} (bc - \eta^\nu ad) = b^{r-2}c^{r-2} - a^{r-2}d^{r-2} = 0$$

where $\eta = \varepsilon_{r-2}$ is a primitive $(r-2)^{\text{th}}$ root of unity. This root exists in $\bar{k}$ since $\operatorname{char} k \nmid r-2$. At least one of the $r-2$ factors must vanish, but the factor for $\nu = 0$ is $bc - ad = -1 \neq 0$.

If $r = 3$ we are done: In this case $\operatorname{char} k = 2$ and the stabilizer is $G_v = N_r(k)$. Moreover k contains non-trivial 3^{rd} roots of unity if and only if k contains the field $\mathbb{F}_4$, the only extension of degree 2 of the prime field $\mathbb{F}_2$. The multiplicative group $\mathbb{F}_4^\times$ is cyclic of order 3. Thus we have proved:

Proposition 20 *Let k be a field of characteristic 2. Then the stabilizer of the homogeneous polynomial $X^3 + Y^3$ in the group $SL_2(k)$ is*

(i) *the extension of the cyclic group T_3 of order 3 by the matrix $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of order 2, isomorphic with the symmetric group $\mathcal{S}_3$, if k contains the field $\mathbb{F}_4$,*

(ii) *the cyclic group of order 2 generated by J otherwise.*

Now we assume that $r > 3$. Then

$$bc = \zeta ad \quad \text{for a } \zeta \in \bar{k} \text{ with } \zeta \neq 1, \zeta^{r-2} = 1.$$

Clearly then $\zeta = bc/ad$ is in k. If k doesn't contain non-trivial $(r-2)^{\text{th}}$ roots of unity we are done with $G_v = T_r(k)$ (resp. $N_r(k)$ if $\operatorname{char} k = 2$). Otherwise for each such ζ we get further elements of G_v:

The determinant condition $ad - bc = 1$ implies $(1 - \zeta)\, ad = 1$, thus

$$d = \frac{1}{a\,(1-\zeta)}\,. \tag{20}$$

Likewise

$$bc = ad - 1 = \frac{1}{1-\zeta} - 1 = \frac{\zeta}{1-\zeta},$$

$$c = \frac{\zeta}{b\,(1-\zeta)}\,. \tag{21}$$

The conditions $d^r - c^r = 1$ and

$$b^r = a^r - 1 \tag{22}$$

imply that

$$\begin{aligned}
\frac{1}{a^r\,(1-\zeta)^r} - \frac{\zeta^r}{b^r\,(1-\zeta)^r} &= 1,\\
b^r - a^r\zeta^r &= a^r b^r (1-\zeta)^r,\\
a^r - 1 - a^r\zeta^r &= a^{2r}\,(1-\zeta)^r - a^r\,(1-\zeta)^r.
\end{aligned}$$

This is a quadratic equation for a^r:

$$(1-\zeta)^r a^{2r} + a^r\,[\zeta^r - 1 - (1-\zeta)^r] + 1 = 0. \tag{23}$$

Here is the disappointing result:

Proposition 21 *Let $r \geq 5$ be odd and $\operatorname{char} k \mid r-1$. Let v be the homogeneous polynomial $X^r + Y^r$. Then the stabilizer G_v of v in the group $G = SL_2(k)$ consists of*

(i) *the subgroup $T_r(k)$ (resp. $N_r(k)$ if $\operatorname{char} k = 2$),*

(ii) *the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $abcd \neq 0$ such that a^r is a solution of (23), b is a solution of (22), and c, d are given by (21) and (20), where ζ is a non-trivial $(r-2)^{th}$ root of unity in k.*

Problem Find more concrete results on the solutions of (20) – (23).

4.4 The stabilizer for $r = 5$

The case of $\operatorname{char} k \nmid r\,(r-1)$, here $\operatorname{char} k \neq 2, 5$, is settled by Proposition 19. The case of $\operatorname{char} k \mid r$, here $\operatorname{char} k = 5$, is reduced to $X+Y$ by the corollary of Proposition 18 and thus settled. For the remaining case $\operatorname{char} k \mid r-1$, here $\operatorname{char} k = 2$, we up to now only have the vague result of Proposition 21. We'll try to make it more concrete. Instead of attacking the equations directly we'll prove as an intermediate step:

Lemma 2 *Let $\operatorname{char} k = 2$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k)$ with $abcd \neq 0$ stabilize $v = X^5 + Y^5$. Then a, b, c, and d are 15^{th} roots of unity.*

Remark: Since the multiplicative group $\mathbb{F}_{16}^{\times}$ has order 15 and is cyclic it is generated by a primitive 15^{th} root of unity $\varepsilon = \varepsilon_{15}$, and $\mathbb{F}_{16} = \mathbb{F}_2[\varepsilon]$ is a field extension of order 4 with $\mathbb{F}_{16} \supset \mathbb{F}_4 \supset \mathbb{F}_2$. The lemma implies that $a, b, c, d \in \mathbb{F}_{16}$.

Proof. In Proposition 21 ζ is a non-trivial 3^{rd} root of unity, thus $\zeta = \varepsilon^5$ or ε^{10}, and $\zeta^2 + \zeta + 1 = 0$, $\zeta^2 = \zeta + 1$, $\zeta^3 = 1$, $\zeta^4 = \zeta$, $\zeta^5 = \zeta^2 = \zeta + 1$. Since

$$(1-\zeta)^5 = 1 - 5\zeta + 10\zeta^2 - 10\zeta^3 + 5\zeta^4 - \zeta^5 = 1 + \zeta + 0 + 0 + \zeta + (\zeta + 1) = \zeta$$

Equation (23) (for $x = a^5$) becomes

$$0 = \zeta x^2 + x\,[\zeta^5 + 1 + \zeta] + 1 = \zeta x^2 + 1,$$

hence $x^2 = 1/\zeta = \zeta^2$, $a^5 = x = \zeta$, and $a^{15} = x^3 = 1$.

Now $b^5 = a^5 + 1 = \zeta + 1 = \zeta^2$, and $b^{15} = 1$. Equations 20 and 21 yield

$$d = \frac{1}{a\,(1-\zeta)} = \frac{1}{a\,\zeta^2}\,, \quad c = \frac{\zeta}{b\,(1-\zeta)} = \frac{\zeta}{b\,\zeta^2}\,,$$

and show that $d^{15} = 1$ and $c^{15} = 1$. $\diamond$

Corollary 1 *For $G = SL_2(\overline{\mathbb{F}_2})$ and $v = X^5 + Y^5$ we have $G_v \subseteq SL_2(\mathbb{F}_{16})$.*

Therefore in the case $abcd \neq 0$ a group element $g \in G_v$ has the form

$$g = \begin{pmatrix} \varepsilon^s & \varepsilon^t \\ \varepsilon^u & \varepsilon^w \end{pmatrix},$$

It stabilizes v if and only if the following five conditions hold:

$$\varepsilon^{s+w} + \varepsilon^{t+u} = 1, \tag{24}$$

$$\varepsilon^{5u} + \varepsilon^{5w} = 1, \quad \varepsilon^{5s} + \varepsilon^{5t} = 1, \tag{25}$$

$$\varepsilon^{s+4u} + \varepsilon^{t+4w} = 0, \quad \varepsilon^{4s+u} + \varepsilon^{4t+w} = 0. \tag{26}$$

The third pair of conditions (26) is equivalent with

$$s + 4u \equiv t + 4w \pmod{15}, \quad 4s + u \equiv 4t + w \pmod{15}.$$

Adding $3s$ to the first equation and substituting from the second one we get

$$3s + t + 4w \equiv 4s + u + 3u \equiv 4t + w + 3u \pmod{15},$$

$$3s + 3w \equiv 3t + 3u \pmod{15},$$

$$s + w \equiv t + u \pmod{5}. \tag{27}$$

For the evaluation of condition (24) (the determinant) we use a lemma:

Lemma 3 *Let ε be a generator of the multiplicative group $\mathbb{F}_{16}^{\times}$. Then the solutions of the equation*

$$\varepsilon^x + \varepsilon^y = 1 \quad \textit{with } y \equiv x \pmod 5$$

are

(i) $x \equiv 5 \pmod{15}$ *and* $y \equiv 10 \pmod{15}$,

(ii) $x \equiv 10 \pmod{15}$ *and* $y \equiv 5 \pmod{15}$.

Proof. Let $y = x + 5z$ and $\alpha = \varepsilon^5$ (thus α is a non-trivial 3rd root of unity). Then $1 = \varepsilon^x \cdot (1+\alpha^z)$, hence $\varepsilon^x \in \mathbb{F}_2[\alpha] = \mathbb{F}_4 = \{0, \alpha, \alpha^2, 1\}$. The assumption $\varepsilon^x = 1$ yields $\varepsilon^y = 1 + \varepsilon^x = 0$, a contradiction. The remaining possibilities are

$$\varepsilon^x = \alpha = \varepsilon^5 \quad \text{and} \quad \varepsilon^y = 1 + \alpha = \alpha^2 = \varepsilon^{10},$$

that is statement (i), or

$$\varepsilon^x = \alpha^2 = \varepsilon^{10} \quad \text{and} \quad \varepsilon^y = 1 + \alpha^2 = \alpha = \varepsilon^5,$$

that is statement (ii). $\diamond$

Exercise Show that the lemma is true even without the assumption $y \equiv x \pmod 5$.

Thus we have two alternative possibilities for the entries of g:

1a $s + w \equiv 5 \pmod{15}$ and $t + u \equiv 10 \pmod{15}$.

1b $s + w \equiv 10 \pmod{15}$ and $t + u \equiv 5 \pmod{15}$.

Applying the lemma to the pair (25) of conditions we get the alternatives "either 2a or 2b", as well as "either 3a or 3b":

2a $u \equiv 1 \pmod 3$ and $w \equiv 2 \pmod 3$.

2b $u \equiv 2 \pmod 3$ and $w \equiv 1 \pmod 3$.

3a $s \equiv 1 \pmod 3$ and $t \equiv 2 \pmod 3$.

3b $s \equiv 2 \pmod 3$ and $t \equiv 1 \pmod 3$.

We have to check eight combinations of 1, 2, and 3. Six of them are contradictory, but the two combinations

$$1\text{a} \wedge 2\text{b} \wedge 3\text{a} \quad \text{and} \quad 1\text{b} \wedge 2\text{a} \wedge 3\text{b}$$

each yield 16 valid solutions. We won't pursue the calculation beyond this point but only present one of these solutions:

$$\begin{pmatrix} \varepsilon & \varepsilon^2 \\ \varepsilon^8 & \varepsilon^4 \end{pmatrix}.$$

Problem Determine the structure of the group G_v for $r = 5$, $\operatorname{char} k = 2$.

Problem Determine the stabilizer of $X^r + Y^r$ for other odd values of $r \geq 7$ with $\operatorname{char} k \mid r - 1$.

4.5 The stabilizer for r even, $r \geq 4$

By the corollary in 4.2 we may assume that $\operatorname{char} k \nmid r$.

The matrix g from (1) transforms $v = X^r + Y^r$ to

$$\begin{aligned} & d^r X^r - r d^{r-1} b X^{r-1} Y + \binom{r}{2} d^{r-2} b^2 X^{r-2} Y^2 - \cdots \quad - r d b^{r-1} X Y^{r-1} + b^r Y^r \\ + \, & c^r X^r - r c^{r-1} a X^{r-1} Y + \binom{r}{2} c^{r-2} a^2 X^{r-2} Y^2 - \cdots \quad - r c a^{r-1} X Y^{r-1} + a^r Y^r. \end{aligned}$$

The conditions that g is in $SL_2(k)$ and fixes v now yield the equations

$$ad - bc = 1 \tag{28}$$

$$d^r + c^r = 1 \tag{29}$$

$$a^r + b^r = 1 \tag{30}$$

$$ac^{r-1} + bd^{r-1} = 0, \tag{31}$$

$$a^{r-1}c + b^{r-1}d = 0. \tag{32}$$

Assuming again that the characteristic of k doesn't divide $r - 1$ we have $\binom{r}{2} \neq 0$ and get the additional necessary condition

$$a^2 c^{r-2} + b^2 d^{r-2} = 0, \tag{33}$$

Multiplying Equation (31) by a we get

$$0 = a^2 c^{r-1} + abd^{r-1} = -b^2 d^{r-2} c + abd^{r-1} = bd^{r-2} \cdot (ad - bc)$$

using Equation (33). Since the determinant is $ad - bc = 1$ we again conclude that

$$b = 0 \quad \text{or} \quad d = 0.$$

First assume $b = 0$. In this case our equations collapse to

$$ad = 1, \quad d^r + c^r = 1, \quad a^r = 1, \quad ac^{r-1} = 0, \quad \ldots$$

Since $a \neq 0$ by the first one, the fourth one gives $c = 0$. By the third one a is an r^{th} root of 1 and d is its inverse. Thus $g = \Delta(\varepsilon)$ with $\varepsilon^r = 1$.

Now assume $b \neq 0$. Then necessarily $d = 0$, and our equations collapse to

$$-bc = 1, \quad c^r = 1, \quad a^r + b^r = 1, \quad ac^{r-1} = 0, \quad \ldots$$

Since $c \neq 0$ by the first one, the fourth one gives $a = 0$. The solutions are: b an r^{th} root of unity, and $c = -1/b$.

Since

$$\begin{pmatrix} 0 & t \\ -1/t & 0 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we have shown (see also [2]):

Proposition 22 *Let $r \geq 4$ be even. Assume that the characteristic of k doesn't divide $r(r-1)$. Then the stabilizer of the homogeneous polynomial $v = X^r + Y^r$ in the group $G = SL_2(k)$ is $G_v = N_r(k)$. If k contains a primitive r^{th} root of 1, then G_v has order $2r$.*

Now for the remaining case where $\operatorname{char} k \mid r-1$, in particular $\operatorname{char} k \neq 2$. Equation (33) breaks down. Instead we use Equation (32), and multiplying Equation (31) by a^{r-2} we get

$$0 = a^{r-1}c^{r-1} + a^{r-2}bd^{r-1} = a^{r-2}bd^{r-1} - b^{r-1}c^{r-2}d = bd\,(a^{r-2}d^{r-2} - b^{r-2}c^{r-2}).$$

One of the factors must vanish: $b = 0$ or $d = 0$ or $a^{r-2}d^{r-2} - b^{r-2}c^{r-2} = 0$.

The case $b = 0$ results in $ad = 1$, $a^r = 1$, $a^{r-1}c = 0$, $c = 0$,

$$g = \Delta(\varepsilon) \quad \text{with } \varepsilon^r = 1.$$

The case $d = 0$ results in $bc = -1$, $c^r = -1$, $b^r = 1/c^r = -1$ (since r is even), $ac^{r-1} = 0$, $a = 0$, $b^r = 1$, contradiction since $\operatorname{char} k \neq 2$.

In the remaining subcase we have $b \neq 0$ and $d \neq 0$, hence even $abcd \neq 0$, but

$$a^{r-2}d^{r-2} - b^{r-2}c^{r-2} = 0.$$

Thus Lemma 1 also holds for $r \geq 4$ even (excluding the characteristic 2 case).

Now in $\bar{k}$ we have the decomposition

$$\prod_{\nu=0}^{r-3}(ad - \eta^\nu bc) = a^{r-2}d^{r-2} - b^{r-2}c^{r-2} = 0$$

where $\eta = \varepsilon_{r-2}$ is a primitive $(r-2)^{\text{th}}$ root of unity. This root exists in $\bar{k}$ since $\operatorname{char} k \nmid r-2$. At least one of the $r-2$ factors must vanish, but the factor for $\nu = 0$ is $ad - bc = 1 \neq 0$. Hence

$$bc = \zeta ad \quad \text{for a } \zeta \in \bar{k} \text{ with } \zeta \neq 1, \zeta^{r-2} = 1.$$

Clearly $\zeta = bc/ad$ is in k. For each such ζ we get further elements of G_v:

The determinant condition $ad - bc = 1$ implies $(1-\zeta)\,ad = 1$, thus

$$d = \frac{1}{a\,(1-\zeta)}\,. \tag{34}$$

Likewise
$$bc = ad - 1 = \frac{1}{1-\zeta} - 1 = \frac{\zeta}{1-\zeta},$$

$$c = \frac{\zeta}{b\,(1-\zeta)}\,. \tag{35}$$

The conditions $d^r + c^r = 1$ and

$$b^r = 1 - a^r \tag{36}$$

imply that

$$\begin{aligned} \frac{1}{a^r\,(1-\zeta)^r} + \frac{\zeta^r}{b^r\,(1-\zeta)^r} &= 1, \\ b^r + a^r\zeta^r &= a^r b^r (1-\zeta)^r, \\ 1 - a^r + a^r\zeta^r &= a^r\,(1-\zeta)^r - a^{2r}\,(1-\zeta)^r. \end{aligned}$$

This is the same quadratic equation for a^r as in (23):

$$(1-\zeta)^r a^{2r} + a^r\,[\zeta^r - 1 - (1-\zeta)^r] + 1 = 0. \tag{37}$$

Here is the result:

Proposition 23 *Let $r \geq 4$ be even and $\operatorname{char} k \mid r-1$. Let v be the homogeneous polynomial $X^r + Y^r$. Then the stabilizer G_v of v in the group $G = SL_2(k)$ consists of*

(i) *the subgroup $T_r(k)$,*

(ii) *the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $abcd \neq 0$ such that a^r is a solution of* (37)*, b is a solution of* (36)*, and c, d are given by* (35) *and* (34)*, where ζ is a non-trivial $(r-2)^{th}$ root of unity in k.*

Problem Find more concrete results on the solutions of (34) – (37).

4.6 The stabilizer for $r = 4$

The case of $\operatorname{char} k \nmid r(r-1)$, here $\operatorname{char} k \neq 2, 3$, is settled by Proposition 22. The case of $\operatorname{char} k \mid r$, here $\operatorname{char} k = 2$, is reduced to $X + Y$ by the corollary of Proposition 18, hence settled.

For the remaining case $\operatorname{char} k \mid r - 1$, here $\operatorname{char} k = 3$, we'll try to make Proposition 23 more concrete. First we'll prove:

Lemma 4 *Let $\operatorname{char} k = 3$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k)$ with $abcd \neq 0$ stabilize $v = X^4 + Y^4$. Then a, b, c, and d are 8^{th} roots of unity.*

Remark: $\mathbb{F}_3$ doesn't contain a primitive 4^{th} root of unity, and a forteriori not an 8^{th} one. But its extension $\mathbb{F}_9$ of degree 2 has a multiplicative group of order 8 that is generated by ε_8 and contains $\varepsilon_4 = \varepsilon_8^2$. The lemma implies that $a, b, c, d \in \mathbb{F}_9$.
Proof. In Proposition 23 necessarily $\zeta = -1$. Furthermore $1 - \zeta = 2 = -1$, $(1-\zeta)^4 = 1$. Equation (37) for $x = a^4$ becomes

$$0 = x^2 + x\,[1 - 1 - 1] + 1 = x^2 - x + 1 = (x+1)^2.$$

Its only solution is $x = -1$, hence $a^4 = -1$, $a^8 = 1$.

From (36) we get $b^4 = 2 = -1$, $b^8 = 1$. From $c = 1/b$ and $d = -1/a$ also $c^8 = d^8 = 1$ $\diamond$

Now the situation is analogous with that in Section 4.4: A group element $g \in G_v$ with $abcd \neq 0$ has the form

$$g = \begin{pmatrix} \varepsilon^s & \varepsilon^t \\ \varepsilon^u & \varepsilon^w \end{pmatrix},$$

where $\varepsilon = \varepsilon_8$, in particular $G_v \subseteq SL_2(\mathbb{F}_9)$.

A similar calculation yields further solutions for $g \in G_v$, for example this one:

$$\begin{pmatrix} \varepsilon & \varepsilon^3 \\ \varepsilon^5 & \varepsilon^3 \end{pmatrix},$$

Problem Determine the structure of the group G_v for $r = 4$, $\operatorname{char} k = 3$.

Problem Determine the stabilizer of $X^r + Y^r$ for other even values of $r \geq 6$ with $\operatorname{char} k \mid r - 1$.

5 Stabilizer and Orbit of $X^rY + XY^r$

The case $r = 0$ was treated in 2.1: The stabilizer is maximal unipotent. The case $r = 1$ is void for $\operatorname{char} k = 2$. The modified case with $u = XY$ was treated in 2.2: The stabilizer is the canonical maximal torus. The case $r = 2$ is in Section 2.3: The stabilizer of $X^2Y + XY^2$ is finite of order 3 for $\operatorname{char} k \neq 2$, of order 6 for $\operatorname{char} k = 2$.

We henceforth assume that $r \geq 2$, and are going to prove:

Theorem 3 *Assume that k is algebraically closed. Let $r \geq 2$ and v be the binary form $v = X^rY + XY^r$ of degree $r + 1$. Then:*

(i) *The stabilizer of v in $G = SL_2(k)$ is finite.*

(ii) *v is*

- *unstable if $r \geq 4$ and $r - 1$ is a power of* $\operatorname{char} k$*, in particular the orbit $G \cdot v$ contains 0 in its closure (type Ib or Ie),*
- *semistable with non-closed orbit if* $\operatorname{char} k = 2$ *and $r = 3$ (type Id),*
- *properly stable otherwise (type Ia).*

(iii) *The orbit map $G \longrightarrow G \cdot v$ is separable if and only if* $\operatorname{char} k \nmid r - 1$.

Statement (i) is proved in Corollary 4 in 1.5. The proof of (ii) follows in 5.1, the proof of (iii) is in 5.2.

The sections 5.3 – 5.5 contain the explicit determination of the finite group G_v over an arbitrary, not necessarily algebraically closed field k in some cases, and highlight the complexity of this task in other cases. A general remark: Let ε be an $(r-1)^{\text{th}}$ root of unity. Since the diagonal matrix $\Delta(t)$, $t \in k^\times$, transforms

$$X^rY \mapsto t^{-r+1}X^rY, \quad XY^r \mapsto t^{r-1}XY^r,$$

$\Delta(\varepsilon)$ stabilizes $v = X^rY + XY^r$. Thus $T_{r-1}(k) \subseteq G_v$ in any case.

5.1 Stability

In the factorization $X^rY + XY^r = XY\,(X^{r-1} + Y^{r-1})$, if $\operatorname{char} k \nmid r - 1$, all the linear factors are simple. Thus v is (properly) stable under the action of SL_2, and the orbit is of type Ia.

In the case of $p = \operatorname{char} k \mid r - 1$ the stability depends on the prime decomposition of $r - 1$. In analogy with 4.2 we decompose $r - 1 = sp^t$ with $t > 0$ and $p \nmid s$ and distinguish two cases:

Case 1, $r-1$ is a power of p:

Then $s = 1$, and all linear factors of $X^{r-1} + Y^{r-1}$ are identical. Since $r-1 \geq p \geq 2$ we have $r \geq 3$. If even $r \geq 4$, then there is a linear factor of multiplicity $r-1 > \frac{r+1}{2}$, hence orbit is unstable, thus contains 0 in its closure.

The case $r = 3$ can occur only for $p = 2$. Then our form $X^3Y + XY^3 = XY\,(X^2 + Y^2)$ has two simple linear factors and one of multiplicity 2. Hence its orbit is semistable, but not stable. Looking at the taxonomy of 1.5 we see that this must be type Id, in particular the orbit is not closed.

Problem (for $r \geq 4$) Decide between the orbit types Ib and Ie.

Case 2, $r-1$ is not a power of p:

Then $s \geq 2$, hence $p^t = \frac{r}{s} \leq \frac{r}{2} < \frac{r+1}{2}$. Our form

$$X^rY + XY^r = XY\,(X^{p^t} + Y^{p^t})^s$$

has two simple linear factors and s ones of multiplicity p^t. Hence v is (properly) stable, and the orbit closed.

The proof of Theorem 3 (ii) is complete.

5.2 The Lie algebra action for $r \geq 2$

The following Proposition implies Theorem 3 (iii).

Proposition 24 *Assume $r \geq 2$. Then the stabilizer of the homogeneous polynomial $v = X^rY + XY^r$ in the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(k)$ is*

(i) $\mathfrak{g}_v = \mathfrak{t}$ *if* $\operatorname{char} k \mid r-1$ *where* $\mathfrak{t}$ *is the subalgebra consisting of the diagonal matrices in $\mathfrak{sl}_2(k)$ (the Lie algebra of the canonical maximal torus), and the orbit map is inseparable.*

(ii) $\mathfrak{g}_v = 0$ *if* $\operatorname{char} k \nmid r-1$*, and the orbit map is separable.*

Proof. The Lie algebra element $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g} = \mathfrak{sl}_2(k)$ transforms $v = X^rY + XY^r$ to

$$-cX^{r+1} - (r-1)\,a\,X^rY - rb\,X^{r-1}Y^2 - rcX^2Y^{r-1} + (r-1)\,a\,XY^r - bY^{r+1},$$

If A annihilates v, then $b = c = 0$, regardless of the characteristic and the value of $r \geq 2$.

If $\operatorname{char} k \mid r-1$ this implies $Av = 0$, whence (i).

Now let $\operatorname{char} k \nmid r-1$. Depending on the value of r there is some overlap of the terms of Av. If $r = 2$ we have a term $-aX^2Y - rcX^2Y$, and since

$c = 0$ for $Av = 0$ we conclude that $a = 0$, hence $A = 0$. If $r = 3$ then $\operatorname{char} k \neq 2$ and we have $2a + 3c = 0$, hence $a = 0$, thus $A = 0$. For $r \geq 4$ there is no overlap, and $(r-1)a = 0$ implies $a = 0$ and $A = 0$. This proves (ii). $\diamond$

5.3 The stabilizer for $r = 3$

We consider the quartic form $v = X^3Y + XY^3$. The matrix g from (1) transforms it to

$$\begin{aligned} &-cd^3X^4 + 3bcd^2X^3Y - 3b^2cdX^2Y^2 + b^3cXY^3 \\ &\qquad + ad^3X^3Y - 3abd^2X^2Y^2 + 3ab^2dXY^3 - ab^3Y^4 \\ &-c^3dX^4 + 3ac^2dX^3Y - 3a^2cdX^2Y^2 + a^3dXY^3 \\ &\qquad + bc^3X^3Y - 3abc^2X^2Y^2 + 3a^2bcXY^3 - a^3bY^4. \end{aligned}$$

The conditions that g is in $SL_2(k)$ and fixes v are equivalent with the equations

$$ad - bc = 1 \tag{38}$$

$$cd^3 + c^3d = 0 \tag{39}$$

$$3bcd^2 + ad^3 + 3ac^2d + bc^3 = 1 \tag{40}$$

$$3\,(b^2cd + abd^2 + a^2cd + abc^2) = 0 \tag{41}$$

$$b^3c + 3ab^2d + a^3d + 3a^2bc = 1 \tag{42}$$

$$ab^3 + a^3b = 0 \tag{43}$$

Equation (39) is equivalent with $cd(d^2 + c^2) = 0$, Equation (43) with $ab(b^2 + a^2) = 0$. Therefore we distinguish three cases: $c = 0$ or $d = 0$ or $c^2 = -d^2$, and in the third case we distinguish between $a = 0$ or $b = 0$ or $b^2 = -a^2$.

If $c = 0$ we get (38) $ad = 1$ and (40) $ad^3 = 1$, hence $d^2 = 1$. Furthermore

- if $\operatorname{char} k \neq 3$, (41) yields $abd^2 = 0$, hence $ab = 0$, and by (42) $a^3d = 1$,
- if $\operatorname{char} k = 3$, also $a^3d = 1$, this time immediately by (42).

Thus $a^2 = 1$, and

- if $\operatorname{char} k \neq 3$, $b = 0$ by (41), hence $g = \pm\mathbf{1}$,
- if $\operatorname{char} k = 3$, by (43) $0 = a^2b^3 + a^4b = b^3 + b = b\,(b^2 + 1)$, yielding the solutions $b = 0$ or (if $k \supseteq \mathbb{F}_9$) $b = \pm i$, with $a = d = \pm 1$.

If $d = 0$, $c \neq 0$, we get (38) $-bc = 1$ and (40) $bc^3 = 1$, hence $c^2 = -1$, $c = \pm i$, $b = -1/c = c$, and

- if $\operatorname{char} k \neq 3$, $abc^2 = 0$ by (41), hence $a = 0$,
- if $\operatorname{char} k = 3$, by (43) $0 = ab^4 + a^3b^2 = a - a^3 = a(1 - a^2)$, yielding the solutions $a = 0$ or $a = \pm 1$.

Therefore g is one of the matrices

- (if $\operatorname{char} k \neq 2, 3$ and $i \in k$) $I = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ or $I^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$,
- (if $\operatorname{char} k = 2$) I,
- (if $\operatorname{char} k = 3$ and $\mathbb{F}_9 \subseteq k$) $I = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ or $I^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ or one of

$$\begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -i \\ -i & 0 \end{pmatrix}.$$

The remaining case is $c^2 = -d^2$ with $c \neq 0$, $d \neq 0$. If $a = 0$ the conditions imply $-bc = 1$ and $b^2cd = 0$, a contradiction. Likewise for $b = 0$ with $a \neq 0$ we get the contradictory equation $a^2cd = 0$.

This leaves us with the case where a, b, c, d are all $\neq 0$, and $c^2 = -d^2$, $b^2 = -a^2$.

If $\operatorname{char} k = 2$, we have $c^2 = d^2$, $b^2 = a^2$, hence $c = d$ and $b = a$, contradicting (38).

If $\operatorname{char} k = 3$, we conclude $c = \pm id$ and $b = \pm ia$ where the signs are different because of $1 = ad - bc = ad \pm ad$. But then by (40) $1 = ad^3 + bc^3 = ad^3 - i^4ad^3 = ad^3 - ad^3 = 0$, contradiction.

Now in the case $\operatorname{char} k \neq 2, 3$ from (40) and (42) we get $-2d^2 = 1$ and $-2a^2 = 1$. This has the four solutions

$$C = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}, \; D = \begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix},$$

$$D^3 = \begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}, \; C^3 = \begin{pmatrix} -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}.$$

Note that $C^2 = D^2 = -\mathbf{1}$, $CI = D^3$, $DI = C$.

This is the result:

Proposition 25 *The stabilizer G_v of the homogeneous polynomial $v = X^3Y + XY^3$ in the group $G = SL_2(k)$ is finite.*

(i) *If $\operatorname{char} k \neq 2, 3$, then G_v is the subgroup H of order 8 generated by the matrices I and D, each of order 4, provided that k contains square roots of -1 and 2, otherwise G_v is the intersection of $H \subseteq SL_2(\bar{k})$ with $SL_2(k)$.*

(ii) *If* $\operatorname{char} k = 2$, *then the stabilizer is the cyclic group of order* 2 *generated by* $I = J$.

(iii) *If* $\operatorname{char} k = 3$, *then the stabilizer is finite of order* 12 *if* $\mathbb{F}_9 \subseteq k$, *of order* 2 *if* $\mathbb{F}_9 \not\subseteq k$.

Problem Determine the group structure of the stabilizer for $\operatorname{char} k \neq 2$.

5.4 The stabilizer for r even, $r \geq 4$

The matrix g from (1) transforms $v = X^rY + XY^r$ to

$$\begin{aligned}
&- cd^rX^{r+1} + rbcd^{r-1}X^rY - \binom{r}{2}b^2cd^{r-2}X^{r-1}Y^2 + \cdots && - b^rcXY^r \\
&\qquad + ad^rX^rY - rabd^{r-1}X^{r-1}Y^2 + \cdots && - rab^{r-1}dXY^r + ab^rY^{r+1} \\
&+ c^rdX^{r+1} - rac^{r-1}dX^rY + \binom{r}{2}a^2c^{r-2}dX^{r-1}Y^2 - \cdots && + a^rdXY^r \\
&\qquad - bc^rX^rY + rabc^{r-1}X^{r-1}Y^2 - \cdots && + ra^{r-1}bcXY^r - a^rbY^{r+1}.
\end{aligned}$$

The conditions that g is in $SL_2(k)$ and fixes v are equivalent with the equations

$$ad - bc = 1 \tag{44}$$

$$c^rd - cd^r = 0 \tag{45}$$

$$rbcd^{r-1} + ad^r - rac^{r-1}d - bc^r = 1 \tag{46}$$

$$\binom{r}{2}a^2c^{r-2}d + rabc^{r-1} - \binom{r}{2}b^2cd^{r-2} - rabd^{r-1} = 0 \tag{47}$$

$$\vdots$$

$$a^rd + ra^{r-1}bc - b^rc - rab^{r-1}d = 1 \tag{48}$$

$$ab^r - a^rb = 0 \tag{49}$$

Equation (45) is equivalent with $cd(c^{r-1} - d^{r-1}) = 0$, equation (49) with $ab(b^{r-1} - a^{r-1}) = 0$. Therefore we distinguish three cases: $c = 0$ or $d = 0$ or $c^{r-1} = d^{r-1}$, and in the third case we distinguish between $a = 0$ or $b = 0$ or $a^{r-1} = b^{r-1}$.

Case 1, $c = 0$**:**

Equations (44), (46), (47), (49) collapse to

$$\begin{aligned}
ad &= 1 \\
ad^r &= 1 \\
rabd^{r-1} &= 0 \\
ab^r - a^rb &= 0
\end{aligned}$$

From the first two of these we conclude $d = 1/a$ and $d^{r-1} = 1$, $a^{r-1} = 1$, thus $a^r = a$, and the fourth equation yields $a\,(b^r - b) = 0$, hence $b^r = b$, thus b is 0 or an $(r-1)^{\text{th}}$ root of unity. If $\operatorname{char} k \nmid r$, the third equation yields $b = 0$. Hence g is in the cyclic subgroup $T_{r-1}(k)$ consisting of the diagonal matrices $\Delta(\varepsilon)$ where ε runs through the $(r-1)^{\text{th}}$ roots of 1 in k. If however $\operatorname{char} k \mid r$, then b could also be an $(r-1)^{\text{th}}$ root of unity, and we get additional solutions of the form $\begin{pmatrix} \varepsilon & \delta \\ 0 & 1/\varepsilon \end{pmatrix}$ with $\delta^{r-1} = 1$. We resolve the situation by a lemma:

Lemma 5 *Let $r \geq 2$ be even, $\operatorname{char} k = p \mid r$, and δ and ε be $(r-1)^{th}$ roots of unity in k. The matrix*

(i) $g_1 = \begin{pmatrix} \varepsilon & \delta \\ 0 & 1/\varepsilon \end{pmatrix} \in SL_2(k)$

(ii) $g_2 = \begin{pmatrix} \varepsilon & 0 \\ \delta & 1/\varepsilon \end{pmatrix} \in SL_2(k)$

(iii) $g_3 = \begin{pmatrix} -\delta & \varepsilon \\ -1/\varepsilon & 0 \end{pmatrix} \in SL_2(k)$

(iv) $g_4 = \begin{pmatrix} 0 & \varepsilon \\ -1/\varepsilon & \delta \end{pmatrix} \in SL_2(k)$

stabilizes the homogeneous polynomial $X^rY + XY^r$ if and only if $p = 2$ and r is a power of p.

Proof. The image of $X^rY + XY^r$ under g_1, g_2, g_3, g_4 is respectively

$$\left(\frac{1}{\varepsilon}\,X - \delta\,Y\right)^r \cdot \varepsilon\,Y + \left(\frac{1}{\varepsilon}\,X - \delta\,Y\right) \cdot (\varepsilon\,Y)^r = \left(\frac{1}{\varepsilon}\,X - \delta\,Y\right)^r \cdot \varepsilon\,Y + X\,Y^r - \delta\varepsilon\,Y^{r+1}$$

$$\left(\frac{1}{\varepsilon}\,X\right)^r (-\delta\,X + \varepsilon\,Y) + \frac{1}{\varepsilon}\,X\,(-\delta\,X + \varepsilon\,Y)^r = -\frac{\delta}{\varepsilon}\,X^{r+1} + X^rY + \frac{1}{\varepsilon}\,X \cdot (-\delta X + \varepsilon\,Y)^r$$

$$(-\varepsilon\,Y)^r \cdot \left(\frac{1}{\varepsilon}\,X - \delta\,Y\right) - \varepsilon\,Y \cdot \left(\frac{1}{\varepsilon}\,X - \delta\,Y\right)^r = X\,Y^r - \delta\varepsilon\,Y^{r+1} - \varepsilon\,Y \cdot \left(\frac{1}{\varepsilon}\,X - \delta\,Y\right)^r$$

$$(\delta\,X - \varepsilon\,Y)^r \left(\frac{1}{\varepsilon}\,X\right) + (\delta\,X - \varepsilon\,Y) \left(\frac{1}{\varepsilon}\,X\right)^r = \frac{1}{\varepsilon}\,X \cdot (\delta\,X - \varepsilon\,Y)^r + \frac{\delta}{\varepsilon}\,X^{r+1} - X^rY$$

where we used $\varepsilon^r = \varepsilon$ and $\delta^r = \delta$.

As first case we assume that r is a power of p, say $r = p^t$ where $t \geq 1$ (and necessarily $p = 2$ since r is even). Then the image of $X^rY + XY^r$ under g_1, g_2, g_3, g_4 is respectively

$$\left(\frac{1}{\varepsilon}\,X^r + \delta\,Y^r\right) \cdot \varepsilon\,Y + X\,Y^r - \delta\varepsilon\,Y^{r+1} = X^rY + \delta\varepsilon\,Y^{r+1} + X\,Y^r - \delta\varepsilon\,Y^{r+1}$$

$$-\frac{\delta}{\varepsilon}X^{r+1} + X^rY + \frac{1}{\varepsilon}X\cdot(\delta X^r + \varepsilon Y^r) = -\frac{\delta}{\varepsilon}X^{r+1} + X^rY + \frac{\delta}{\varepsilon}X^{r+1} + XY^r$$

$$XY^r - \delta\varepsilon Y^{r+1} - \varepsilon Y\cdot\left(\frac{1}{\varepsilon}X^r + \delta Y^r\right) = XY^r - \delta\varepsilon Y^{r+1} - X^rY - \delta\varepsilon Y^{r+1}$$

$$\frac{1}{\varepsilon}X\cdot(\delta X^r + \varepsilon Y^r) + \frac{\delta}{\varepsilon}X^{r+1} - X^rY = \frac{\delta}{\varepsilon}X^{r+1} + XY^r + \frac{\delta}{\varepsilon}X^{r+1} - X^rY$$

In all four cases this is $= X^rY + XY^r$ (remember $\operatorname{char} k = 2$).

As second case we assume that r is not a power of p, say $r = sp^t$ where $s \geq 2$ and $p \nmid s$, and $t \geq 1$. Then the image of $X^rY + XY^r$ under g_1 is

$$\begin{aligned}
&= \left(\frac{1}{\varepsilon^{p^t}}X^{p^t} \pm \delta^{p^t}Y^{p^t}\right)^s \cdot \varepsilon Y + XY^r - \delta\varepsilon Y^{r+1}\\
&= \left(\frac{1}{\varepsilon}X^r \pm s\,\frac{\delta^{p^t}}{\varepsilon^{p^t(s-1)}}X^{p^t(s-1)}Y^{p^t} \pm \cdots\right)\cdot \varepsilon Y + XY^r - \delta\varepsilon Y^{r+1}\\
&= X^rY \pm s\eta X^{p^t(s-1)}Y^{p^t+1} \pm \cdots + XY^r - \delta\varepsilon Y^{r+1}
\end{aligned}$$

where η is an $(r-1)^{\text{th}}$ root of unity, a contradiction since $s \neq 0$ in k. The action of g_3 yields the same contradictory term, whereas the actions of g_2 and g_4 each yield the non-zero term

$$\pm s\eta X^{p^t+1}Y^{p^t(s-1)}$$

that yields an analogous contradiction. $\diamond$

Case 2, $d = 0$**,** $c \neq 0$**:**

From Equations (44) and (46) – (49) we get

$$\begin{aligned}
-bc &= 1\\
-bc^r &= 1\\
rabc^{r-1} &= 0\\
ra^{r-1}bc - b^rc &= 1\\
ab^r - a^rb &= 0
\end{aligned}$$

The first two yield $b = -1/c$, $c^{r-1} = 1$, hence $b^{r-1} = -1$ and $b^r = -b$. The last one yields $a^r + a = 0$, hence $a = 0$ or an $(r-1)^{\text{th}}$ root of -1.

Now assume that $\operatorname{char} k \nmid r$ (in particular $\operatorname{char} k \neq 2$). Then the third equation yields $a = 0$, and then the fourth one, $-b^rc = 1$. Substituting $c = -1/b$ we get the contradiction $b^{r-1} = 1$ (since $\operatorname{char} k \neq 2$).

If $\operatorname{char} k \mid r$, then also $b^{r-1} = -1$ and $b^{r-1} = 1$, a contradiction if $\operatorname{char} k \neq 2$. But if $\operatorname{char} k = 2$, we get additional solutions of the form

$\begin{pmatrix} \delta & \varepsilon \\ 1/\varepsilon & 0 \end{pmatrix}$ with $\varepsilon^{r-1} = 1$, and $\delta = 0$ or $\delta^{r-1} = 1$. In the first case we find the solutions $g = \Delta(\varepsilon)\, J$. In the second case we again resolve the situation by Lemma 5: These additional solutions, of type g_3, stabilize $X^rY + X\,Y^r$ if and only if r is a power of $\operatorname{char} k = 2$.

Case 3, $c^{r-1} = d^{r-1}$**,** $d \neq 0$**,** $c \neq 0$**:**

We have $c = \varepsilon d$ with $\varepsilon^{r-1} = 1$, hence also $c^r = \varepsilon d^r$ and $c^{r-2} = \frac{1}{\varepsilon} d^{r-2}$.

Equations (44) and (46) boil down to

$$ad - \varepsilon bd = 1, \quad \text{or} \quad d\,(a - \varepsilon b) = 1$$

$$1 = r\varepsilon bd^r + ad^r - rad^r - \varepsilon bd^r = (r-1)\,(\varepsilon b - a)\,d^r = -(r-1)\,d^{r-1}$$

In the case $\operatorname{char} k \mid r-1$ this is a contradiction that settles this case. Henceforth we may asssume that $\operatorname{char} k \nmid r-1$, and continue with equations (47) – (49)

$$0 = \binom{r}{2} \left[a^2 \frac{1}{\varepsilon} d^{r-1} - b^2 \varepsilon d^{r-1}\right] = \frac{r\,(r-1)}{2} \cdot d^{r-1} \left[\frac{a^2}{\varepsilon} - b^2 \varepsilon\right] \text{ or}$$

$$\frac{r}{2} \cdot \left[a^2 - b^2 \varepsilon^2\right] = 0$$

$$1 = a^r d + r\varepsilon a^{r-1} bd - \varepsilon b^r d - rab^{r-1} d = d \cdot \left[a^{r-1}(a + r\varepsilon b) - b^{r-1}(\varepsilon b + ra)\right]$$

$$ab\,(b^{r-1} - a^{r-1}) = 0$$

We proceed by looking at a and b.

Case 3a, $c^{r-1} = d^{r-1}$**,** $d \neq 0$**,** $c \neq 0$**,** $a = 0$**:**

From (44) we have $-\varepsilon bd = -bc = 1$, hence $bd = -1/\varepsilon$, in particular $b \neq 0$. Equation (47) yields $\frac{r}{2}\, b^2 \varepsilon = 0$, a contradiction except when $\operatorname{char} k \mid r$.

If $\operatorname{char} k \mid r$ equation (46) yields $d^{r-1} = 1$. Then $b = -1/\varepsilon d$, $b^{r-1} = -1$. But (48) says $-b^r c = 1$, thus $bc = 1$, a contradiction if $\operatorname{char} k \neq 2$. But if $\operatorname{char} k = 2$ we get additional solutions of the form $\begin{pmatrix} 0 & \frac{1}{\varepsilon d} \\ \varepsilon d & d \end{pmatrix}$. Case (iv) of Lemma 5 applies: The additional solutions stabilize $X^rY + X\,Y^r$ if and only if r is a power of $\operatorname{char} k = 2$.

Case 3b, $c^{r-1} = d^{r-1}$**,** $d \neq 0$**,** $c \neq 0$**,** $b = 0$**,** $a \neq 0$**:**

Equation (47) yields $\binom{r}{2} a^2 c^{r-2} d = 0$, a contradiction except when $\operatorname{char} k \mid r$.

From (44) we have $ad = 1$, hence $d = 1/a$. If $\operatorname{char} k \mid r$ equation (46) again yields $d^{r-1} = 1$, or $a^{r-1} = 1$, resulting in additional solutions of the form $\begin{pmatrix} a & 0 \\ \varepsilon/a & 1/a \end{pmatrix}$. Case (ii) of Lemma 5 applies: The additional solutions stabilize $X^rY + X\,Y^r$ if and only if r is a power of $\operatorname{char} k = 2$.

Case 3c, $c^{r-1} = d^{r-1}$, $d \neq 0$, $c \neq 0$, $a^{r-1} = b^{r-1}$, $a \neq 0$, $b \neq 0$:

Then $b = \delta a$ with $\delta^{r-1} = 1$. From (44) we get $ad\,(1-\delta\varepsilon) = 1$, in particular $(1-\delta\varepsilon) \neq 0$. Equation (47) yields

$$\binom{r}{2} a^2 d^{r-1} \left(\frac{1}{\varepsilon} - \delta^2\varepsilon\right) = 0, \quad \text{or}$$

$$\frac{r}{2} \cdot (1-\delta^2\varepsilon^2) = \frac{r}{2} \cdot (1-\delta\varepsilon)(1+\delta\varepsilon) = 0$$

If $\operatorname{char} k \nmid r$ (in particular $\operatorname{char} k \neq 2$) we conclude $1+\delta\varepsilon = 0$, thus $\delta = -1/\varepsilon$. This yields the contradiction $\delta^{r-1} = -1/\varepsilon^{r-1} = -1$.

We are left with the case $\operatorname{char} k \mid r$. Then from (46) we get $d^{r-1} = 1$, and from (48) we get

$$1 = a^r d\,(1-\delta\varepsilon) = a^{r-1}.$$

Thus a, b, c, and d are $(r-1)^{\text{th}}$ roots of unity, and g has the form

$$g = \begin{pmatrix} \varepsilon_{r-1}^s & \varepsilon_{r-1}^t \\ \varepsilon_{r-1}^u & \varepsilon_{r-1}^w \end{pmatrix}. \tag{50}$$

(If $r = 2$ we are in characteristic 2 and this solution set is empty.)

Problem Find necessary and sufficient conditions that this matrix stabilzes $X^rY + X\,Y^r$ in the case $\operatorname{char} k \mid r$.

The result:

Collecting these calculations together we get an extension of another result from [2]:

Theorem 4 *Let $r \geq 4$ be even. Then the stabilizer G_v of the homogeneous polynomial $v = X^rY + XY^r$ in the group $G = SL_2(k)$ is finite and contains the cyclic subgroup $T_{r-1}(k)$ consisting of the diagonal matrices $\Delta(\varepsilon)$ where ε runs through the $(r-1)$-th roots of 1 in k. More precisely:*

(i) *If $\operatorname{char} k \nmid r$, then $G_v = T_{r-1}(k)$.*

(ii) *If $\operatorname{char} k \mid r$, $\operatorname{char} k \neq 2$, then G_v consists of $T_{r-1}(k)$ and additional elements of the form (50).*

(iii) *If $\operatorname{char} k = 2$ but r is not a power of 2, then $G_v = N_{r-1}(k)$.*

(iv) *If $\operatorname{char} k = 2$ and r is a power of 2, then G_v consists of $N_{r-1}(k)$ and the matrices of types g_1, g_2, g_3, g_4 from Lemma 5.*

In [2] this result is erroneously stated also for $r = 2$ (in characteristic 0). However the case $r = 2$ is different as shown in 2.3.

5.5 The stabilizer for r odd, $r \geq 5$

The matrix g from (1) transforms $v = X^rY + XY^r$ to

$$\begin{aligned}
&- cd^rX^{r+1} + rbcd^{r-1}X^rY - \binom{r}{2}b^2cd^{r-2}X^{r-1}Y^2 + \cdots && + b^rcXY^r \\
&\qquad + ad^rX^rY - rabd^{r-1}X^{r-1}Y^2 + \cdots && + rab^{r-1}dXY^r - ab^rY^{r+1} \\
&- c^rdX^{r+1} + rac^{r-1}dX^rY - \binom{r}{2}a^2c^{r-2}dX^{r-1}Y^2 - \cdots && + a^rdXY^r \\
&\qquad + bc^rX^rY - rabc^{r-1}X^{r-1}Y^2 - \cdots && + ra^{r-1}bcXY^r - a^rbY^{r+1}.
\end{aligned}$$

(Compared with Section 5.4 only some signs changed.) The conditions that g is in $SL_2(k)$ and fixes v are equivalent with the equations

$$ad - bc = 1 \tag{51}$$

$$c^rd + cd^r = 0 \tag{52}$$

$$rbcd^{r-1} + ad^r + rac^{r-1}d + bc^r = 1 \tag{53}$$

$$\binom{r}{2}a^2c^{r-2}d + rabc^{r-1} + \binom{r}{2}b^2cd^{r-2} + rabd^{r-1} = 0 \tag{54}$$

$$\vdots$$

$$a^rd + ra^{r-1}bc + b^rc + rab^{r-1}d = 1 \tag{55}$$

$$ab^r + a^rb = 0 \tag{56}$$

Equation (52) is equivalent with $cd(c^{r-1} + d^{r-1}) = 0$, equation (56) with $ab(b^{r-1} + a^{r-1}) = 0$. Therefore we distinguish three cases: $c = 0$ or $d = 0$ or $c^{r-1} = -d^{r-1}$, and in the third case we distinguish between $a = 0$ or $b = 0$ or $b^{r-1} = -a^{r-1}$.

Case 1, $c = 0$:

Equations (51), (53), (54), (56) collapse to

$$\begin{aligned}
ad &= 1 \\
ad^r &= 1 \\
rabd^{r-1} &= 0 \\
ab^r + a^rb &= 0
\end{aligned}$$

From the first two of these we conclude $d = 1/a$ and $d^{r-1} = 1$, $a^{r-1} = 1$, thus $a^r = a$, and the fourth equation yields $a\,(b^r + b) = 0$, hence $b^r = -b$, thus b is 0 or an $(r-1)^{\text{th}}$ root of -1. If $\operatorname{char} k \nmid r$, the third equation yields $b = 0$. Hence $g \in T_{r-1}(k)$. If however $\operatorname{char} k \mid r$, then b could also be an $(r-1)^{\text{th}}$ root of -1, and we get additional solutions of the form $\begin{pmatrix} \varepsilon & \delta \\ 0 & 1/\varepsilon \end{pmatrix}$

with $\delta^{r-1} = -1$. We resolve the situation by an analogue of Lemma 5—these additional solutions stabilize X^rY+XY^r if and only if r is a power of char k:

Lemma 6 *Let $r \geq 3$ be odd,* char $k = p \mid r$*. The matrix*

(i) $g_1 = \begin{pmatrix} \varepsilon & \delta \\ 0 & 1/\varepsilon \end{pmatrix} \in SL_2(k)$ *with* $\varepsilon^{r-1} = 1$, $\delta^{r-1} = -1$,

(ii) $g_2 = \begin{pmatrix} \varepsilon & 0 \\ \delta & 1/\varepsilon \end{pmatrix} \in SL_2(k)$ *with* $\varepsilon^{r-1} = 1$, $\delta^{r-1} = -1$,

(iii) $g_3 = \begin{pmatrix} \delta & \varepsilon \\ -1/\varepsilon & 0 \end{pmatrix} \in SL_2(k)$ *with* $\varepsilon^{r-1} = -1$, $\delta^{r-1} = 1$,

(iv) $g_4 = \begin{pmatrix} 0 & \varepsilon \\ -1/\varepsilon & \delta \end{pmatrix} \in SL_2(k)$ *with* $\varepsilon^{r-1} = -1$, $\delta^{r-1} = 1$,

stabilizes the homogeneous polynomial X^rY+XY^r if and only if r is a power of p.

Proof. The image of $X^rY + XY^r$ under g_1, g_2, g_3, g_4 is respectively

$$\left(\frac{1}{\varepsilon}X - \delta Y\right)^r \cdot \varepsilon Y + \left(\frac{1}{\varepsilon}X - \delta Y\right) \cdot (\varepsilon Y)^r = \left(\frac{1}{\varepsilon}X - \delta Y\right)^r \cdot \varepsilon Y + XY^r - \delta\varepsilon Y^{r+1}$$

$$\left(\frac{1}{\varepsilon}X\right)^r (-\delta X + \varepsilon Y) + \frac{1}{\varepsilon}X(-\delta X + \varepsilon Y)^r = -\frac{\delta}{\varepsilon}X^{r+1} + X^rY + \frac{1}{\varepsilon}X \cdot (-\delta X + \varepsilon Y)^r$$

$$(-\varepsilon Y)^r \cdot \left(\frac{1}{\varepsilon}X + \delta Y\right) - \varepsilon Y \cdot \left(\frac{1}{\varepsilon}X + \delta Y\right)^r = XY^r + \delta\varepsilon Y^{r+1} - \varepsilon Y \cdot \left(\frac{1}{\varepsilon}X + \delta Y\right)^r$$

$$(\delta X - \varepsilon Y)^r \left(\frac{1}{\varepsilon}X\right) + (\delta X - \varepsilon Y)\left(\frac{1}{\varepsilon}X\right)^r = \frac{1}{\varepsilon}X \cdot (\delta X - \varepsilon Y)^r - \frac{\delta}{\varepsilon}X^{r+1} + X^rY$$

where we used that $\varepsilon^r = \pm\varepsilon$ and $\delta^r = \pm\delta$.

As first case we assume that r is a power of p, say $r = p^t$ where $t \geq 1$. Then the image of $X^rY + XY^r$ under g_1, g_2, g_3, g_4 is respectively

$$\left(\frac{1}{\varepsilon}X^r + \delta Y^r\right) \cdot \varepsilon Y + XY^r - \delta\varepsilon Y^{r+1} = X^rY + \delta\varepsilon Y^{r+1} + XY^r - \delta\varepsilon Y^{r+1}$$

$$-\frac{\delta}{\varepsilon}X^{r+1} + X^rY + \frac{1}{\varepsilon}X \cdot (\delta X^r + \varepsilon Y^r) = -\frac{\delta}{\varepsilon}X^{r+1} + X^rY + \frac{\delta}{\varepsilon}X^{r+1} + XY^r$$

$$XY^r + \delta\varepsilon Y^{r+1} - \varepsilon Y \cdot \left(-\frac{1}{\varepsilon}X^r + \delta Y^r\right) = XY^r + \delta\varepsilon Y^{r+1} + X^rY - \delta\varepsilon Y^{r+1}$$

$$\frac{1}{\varepsilon}X \cdot (\delta X^r + \varepsilon Y^r) - \frac{\delta}{\varepsilon}X^{r+1} + X^rY = \frac{\delta}{\varepsilon}X^{r+1} + XY^r - \frac{\delta}{\varepsilon}X^{r+1} + X^rY$$

In all four cases this is $= X^rY + XY^r$.

As second case we assume that r is not a power of p, say $r = sp^t$ where $s \geq 2$ and $p \nmid s$, and $t \geq 1$, and s and p are odd. Then the image of $X^rY + XY^r$ under g_1 is

$$\begin{aligned}
&= \left(\frac{1}{\varepsilon^{p^t}}\,X^{p^t} - \delta^{p^t}Y^{p^t}\right)^s \cdot \varepsilon\, Y + X\,Y^r - \delta\varepsilon\, Y^{r+1} \\
&= \left(\frac{1}{\varepsilon}\,X^r - s\,\frac{\delta^{p^t}}{\varepsilon^{p^t(s-1)}}X^{p^t(s-1)}Y^{p^t} + \cdots\right) \cdot \varepsilon\, Y + X\,Y^r - \delta\varepsilon\, Y^{r+1} \\
&= X^rY - s\eta X^{p^t(s-1)}Y^{p^t+1} + \cdots + X\,Y^r - \delta\varepsilon\, Y^{r+1}
\end{aligned}$$

where $\eta \neq 0$, a contradiction since $s \neq 0$ in k. The action of g_2, g_3, and g_4 yield the same contradictory term $s\eta X^{p^t(s-1)}Y^{p^t+1}$ or $s\eta X^{p^t(s-1)+1}Y^{p^t}$. $\diamond$

Case 2, $d = 0$, $c \neq 0$:

From Equations (51) and (53) – (56) we get

$$\begin{aligned}
-bc &= 1 \\
bc^r &= 1 \\
rabc^{r-1} &= 0 \\
ra^{r-1}bc + b^rc &= 1 \\
ab^r + a^rb &= 0
\end{aligned}$$

The first two yield $b = -1/c$, $c^{r-1} = -1$, hence $b^{r-1} = (-1/c)^{r-1} = 1/c^{r-1} = -1$ and $b^r = -b$. The last one yields $b\,(a^r - a) = 0$, hence $a = 0$ or an $(r-1)^{\text{th}}$ root of 1. For $a = 0$ we get the solutions $\begin{pmatrix} 0 & \varepsilon \\ -1/\varepsilon & 0 \end{pmatrix} = \Delta(\varepsilon)J$ with $\varepsilon^{r-1} = -1$.

Now assume that $\operatorname{char} k \nmid r$. Then the third equation yields $a = 0$, and we are done.

If however $\operatorname{char} k \mid r$, then we get additional solutions of the form $\begin{pmatrix} \delta & \varepsilon \\ -1/\varepsilon & 0 \end{pmatrix}$ with $\varepsilon^{r-1} = -1$, $\delta^{r-1} = 1$. Again we resolve the situation by Lemma 6: These additional solutions stabilize $X^rY + X\,Y^r$ if and only if r is a power of $\operatorname{char} k$.

Case 3, $c^{r-1} = -d^{r-1}$, $d \neq 0$, $c \neq 0$:

We have $c = \varepsilon d$ with $\varepsilon^{r-1} = -1$, hence also $c^r = -\varepsilon d^r$ and $c^{r-2} = -\frac{1}{\varepsilon}d^{r-2}$.

Equations (51) and (53) boil down to

$$ad - \varepsilon bd = 1, \quad \text{or} \quad d\,(a - \varepsilon b) = 1$$

$$1 = r\varepsilon bd^r + ad^r - rad^r - \varepsilon bd^r = (r-1)\,(\varepsilon b - a)\,d^r = -(r-1)\,d^{r-1}$$

In the case $\operatorname{char} k \mid r-1$ this is a contradiction that settles this case.

Henceforth we may asssume that $\operatorname{char} k \nmid r-1$, and continue with equations (54) – (56)

$$0 = \binom{r}{2}\left[b^2\varepsilon d^{r-1} - a^2\frac{1}{\varepsilon}d^{r-1}\right] = \binom{r}{2}\cdot d^{r-1}\left[b^2\varepsilon - \frac{a^2}{\varepsilon}\right] \quad \text{or}$$

$$\binom{r}{2}\cdot\left[b^2\varepsilon^2 - a^2\right] = 0$$

$$1 = a^r d + r\varepsilon a^{r-1}bd + \varepsilon b^r d + rab^{r-1}d = d\cdot\left[a^{r-1}(a + r\varepsilon b) + b^{r-1}(\varepsilon b + ra)\right]$$

$$ab\,(b^{r-1} + a^{r-1}) = 0$$

We proceed by looking at a and b.

Case 3a, $c^{r-1} = -d^{r-1}$, $d \neq 0$, $c \neq 0$, $a = 0$:

From (51) we have $-\varepsilon bd = -bc = 1$, hence $bd = -1/\varepsilon$, in particular $b \neq 0$. Equation (54) yields $\binom{r}{2}b^2\varepsilon = 0$, a contradiction except when $\binom{r}{2} = 0$. This exception occurs only for $\operatorname{char} k \mid r$ (since $\operatorname{char} k = 2$ was already excluded by $\operatorname{char} k \nmid r-1$).

If $\operatorname{char} k \mid r$ equation (53) yields $c^{r-1} = -1$, hence $d^{r-1} = 1$, resulting in additional solutions of the form $\begin{pmatrix} 0 & \varepsilon \\ -1/\varepsilon & \delta \end{pmatrix}$ with $\varepsilon^{r-1} = -1$, $\delta^{r-1} = 1$. Case (iv) of Lemma 6 applies: The additional solutions stabilize $X^rY + XY^r$ if and only if r is a power of $\operatorname{char} k$.

Case 3b, $c^{r-1} = -d^{r-1}$, $d \neq 0$, $c \neq 0$, $b = 0$, $a \neq 0$:

From (51) we have $ad = 1$, hence $d = 1/a$. Equation (54) yields $\binom{r}{2}a^2 = 0$, a contradiction except when $\operatorname{char} k \mid r$.

If $\operatorname{char} k \mid r$ equation (53) again yields $a^{r-1} = 1$, resulting in additional solutions of the form $\begin{pmatrix} \varepsilon & 0 \\ \delta & 1/\varepsilon \end{pmatrix}$ with $\varepsilon^{r-1} = 1$, $\delta^{r-1} = -1$. Case (ii) of Lemma 6 applies: The additional solutions stabilize $X^rY + XY^r$ if and only if r is a power of $\operatorname{char} k$.

Case 3c, $c^{r-1} = -d^{r-1}$, $d \neq 0$, $c \neq 0$, $b^{r-1} = -a^{r-1}$, $a \neq 0$, $b \neq 0$:

Then $b = \delta a$ with $\delta^{r-1} = -1$. From (51) we get $ad\,(1 - \delta\varepsilon) = 1$, in particular $1 - \delta\varepsilon \neq 0$. Equation (53) yields

$$1 = -(r-1)\,d^{r-1},$$

equation (54)

$$\binom{r}{2} \cdot \left(\delta^2\varepsilon^2 a^2 - a^2\right) = 0,$$

and equation (55)

$$1 = d\left[\delta\varepsilon a(r-1)a^{r-1} + a(1-r)a^{r-1}\right] = da^r\,(r-1)\,(\delta\varepsilon - 1),$$

thus, using $ad\,(1-\delta\varepsilon) = 1$, $1 = -a^{r-1}(r-1)$, hence

$$a^{r-1} = d^{r-1} = -\frac{1}{r-1}.$$

Case 3c – $\operatorname{char} k \nmid r$

If $\operatorname{char} k \nmid r$ we conclude that $0 = \delta^2\varepsilon^2 - 1 = (\delta\varepsilon-1)(\delta\varepsilon+1)$, hence $\delta\varepsilon+1 = 0$, thus $\delta = -1/\varepsilon$, and $1-\delta\varepsilon = 2$, hence $2ad = 1$.

We get the solution

$$d = \frac{1}{2a}, \quad b = -\frac{a}{\varepsilon}, \quad c = \frac{\varepsilon}{2a}$$

for $a \in k$ with $a^{r-1} = -1/(r-1)$, leading to the matrix

$$(57) \qquad \begin{pmatrix} a & -\frac{a}{\varepsilon} \\ \frac{\varepsilon}{2a} & \frac{1}{2a} \end{pmatrix} \quad \text{with } a^{r-1} = -1/(r-1),\ \varepsilon^{r-1} = -1.$$

Whether the matrix (57) really defines a solution however depends on a side condition: From $d = 1/2a$ we conclude that

$$-\frac{1}{d-1} = d^{r-1} = \frac{1}{2^{r-1}a^{r-1}} = -\frac{r-1}{2^{r-1}},$$

or $2^{r-1} = (r-1)^2$ in k. Both sides of this equation lie in the prime field, thus the side condition is: $r-1$ is a solution of the congruence

$$(58) \qquad 2^x \equiv x^2 \pmod{\operatorname{char} k}.$$

Case 3c – $\operatorname{char} k \mid r$

We are left with the case $\operatorname{char} k \mid r$. Then from (53) we get $d^{r-1} = 1$, and from (55) we get $a^{r-1} = 1$. Thus a and d are $(r-1)^{\text{th}}$ roots of unity, b and c are $(r-1)^{\text{th}}$ roots of -1, and g has the form

$$(59) \qquad g = \begin{pmatrix} \varepsilon_l^s & \varepsilon_l^t \\ \varepsilon_l^u & \varepsilon_l^w \end{pmatrix} \quad \text{with } l = 2r-2,\ s \text{ and } w \text{ even}.$$

Problem Find necessary and sufficient conditions that this matrix stabilzes $X^rY + X\,Y^r$ in the case $\operatorname{char} k \mid r$.

Theorem 5 *Let $r \geq 5$ be odd. Then the stabilizer G_v of the homogeneous polynomial $v = X^rY + XY^r$ in the group $G = SL_2(k)$ is finite and contains the dihedral subgroup $N_{r-1}(k)$. More precisely:*

(i) *If* $\operatorname{char} k \nmid r\,(r-1)$*, or if* $\operatorname{char} k \mid r$ *but r is not a power of* $\operatorname{char} k$*, then*

- $G_v = N_{r-1}(k)$ *if* $\operatorname{char} k$ *doesn't fulfill* (58),
- G_v *consists of* $N_{r-1}(k)$ *and all matrices* (57) *if* $\operatorname{char} k$ *fulfills* (58).

(ii) *If* $\operatorname{char} k \mid r-1$*, then* $G_v = N_{r-1}(k)$.

(iii) *If r is a power of* $\operatorname{char} k$*, then*

- G_v *consists of* $N_{r-1}(k)$ *and all matrices of types* g_1, g_2, g_3, g_4 *from Lemma 6 if* $\operatorname{char} k$ *doesn't fulfill* (58),
- G_v *consists of* $N_{r-1}(k)$*, all matrices of types* g_1, g_2, g_3, g_4 *from Lemma 6, and all matrices* (57) *if* $\operatorname{char} k$ *fulfills* (58).

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