The Number of Orbits of Finite Group Actions

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Let a group G act on a set M. Denote by M^g the set of fixed points of the group element $g \in G$, and by G_x, the stabilizer of the element $x \in M$. The following is known as "Burnside's lemma" but was known to Cauchy already in 1845.

Proposition 1 *Let the finite group G act on the finite set M. Then the number of orbits equals*

$$\frac{1}{\#G} \cdot \sum_{g \in G} \#M^g.$$

Proof. The preimage of the diagonal $\Delta = \{(x,x) \mid x \in M\}$ under the map

$$\Phi\colon G \times M \longrightarrow M \times M, \quad (g,x) \mapsto (g \cdot x, x),$$

$$\Phi^{-1}(\Delta) = \{(g,x) \mid g \in G, x \in M, g \cdot x = x\},$$

decomposes in two different ways:

$$\Phi^{-1}(\Delta) = \dot{\bigcup_{g \in G}} \{(g,x) \mid x \in M^g\} = \dot{\bigcup_{x \in M}} \Phi^{-1}(x,x) = \dot{\bigcup_{x \in M}} \{(g,x) \mid g \in G_x\}.$$

Counting the number of elements using the first decomposition yields

$$\#\Phi^{-1}(\Delta) = \sum_{g \in G} \#M^g,$$

counting using the second decomposition yields

$$\#\Phi^{-1}(\Delta) = \sum_{x \in M} \#G_x = \#G \cdot \sum_{x \in M} \frac{1}{\#(G \cdot x)}.$$

In the last sum each orbit $G \cdot x$ contributes the partial sum $\sum_{y \in G \cdot x} 1/\#(G \cdot x) = 1$. Therefore the entire sum is $\#G$ times the number of orbits. $\diamond$