The Hilbert-Mumford Criterion

Klaus Pommerening
Johannes-Gutenberg-Universität
Mainz, Germany

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The notions of stability and related notions apply for actions of algebraic groups on algebraic varieties, but are relevant almost only for actions of reductive groups. In this text we confine ourselves to linear actions, a case where these notions are especially easy to understand.

1 Stability for Linear Actions

Definition Let G be an affine algebraic group over an algebraically closed field k. Let V be a finite-dimensional rational G-module. A point $x \in V$ is called

unstable if 0 is in the closure of the orbit $G \cdot x$, i. e. $0 \in \overline{G \cdot x}$,

semistable if x is not unstable, i. e. if $0 \notin \overline{G \cdot x}$,

stable if $x \neq 0$ and the orbit $G \cdot x$ is closed and of maximal dimension (among all orbits),

properly stable if $x \neq 0$, the orbit $G \cdot x$ is closed, and the stabilizer G_x is finite.

We denote the sets of unstable, semistable, stable, or properly stable points by

$$V_u, \quad V_{ss}, \quad V_s, \quad V_s^{(0)} \quad \text{respectively.}$$

We call the action of G on V **properly stable** (**stable**, **semistable**) if there are properly stable (stable, semistable) points in V, in other words if $V_s^{(0)} \neq \emptyset$ ($V_s \neq \emptyset$, $V_{ss} \neq \emptyset$). We call the action **unstable** if all points are unstable.

Remarks

1. $V = V_u \,\dot{\cup}\, V_{ss}$.
2. $V_s^{(0)} \subseteq V_s \subseteq V_{ss}$.

3. $V_s^{(0)} = V_s$ or $\emptyset$, depending on whether

$$\max \dim\{G \cdot x \mid x \in V\} = \dim G \quad \text{or} \quad < \dim G.$$

In other words: If there are properly stable points, then all stable are properly stable.

4. If the action is properly stable, or $V_s^{(0)} \neq \emptyset$, then $\operatorname{trdeg} K(V)^G = \dim V - \dim G$, in particular $\dim V \geq \dim G$.

5. If x is unstable, then $\rho(x) = \rho(0)$ for every morphism $\rho\colon V \longrightarrow Y$ that is constant on the orbits. In particular this holds for every invariant $\rho \in \mathcal{O}(V)^G$.

Problem Let G be almost simple, and let V be an irreducible G-module with $\dim V \geq \dim G$, but $V \ncong \mathfrak{g} = \operatorname{Lie}(G)$ as G-module. Is the action of G on V properly stable? (In characteristic 0 this follows by a stupid case-by-case inspection.)

Examples For $G = SL_2$ and $V = R_d$, the vector space of forms of degree d we consider (as $x \in V$) a form $F \in R_d$. We'll prove in Section 4:

- F is unstable if and only if F has a linear factor of multiplicity $> \frac{d}{2}$.
- F is semistable if and only if all linear factors of F have multiplicity $\leq \frac{d}{2}$.
- (For $d \geq 3$) F is stable if and only if F has only linear factors of multiplicity $< \frac{d}{2}$. In this case F is even properly stable.

2 Stability for Reductive Groups

Now we assume that G is reductive. Then we know that the invariant algebra $\mathcal{O}(V)^G$ is finitely generated and defines a "good" quotient $\pi : V \longrightarrow V/G$, that is a morphism with the properties

- $\mathcal{O}(V/G) = \mathcal{O}(V)^G$.
- π is constant on the orbits.
- π has the universal property for morphisms that are constant on the orbits, in other words, it is a categorial quotient.
- π separates the closed orbits.

Remarks Let G be reductive, and V be a rational G-module.

1. $V_u = \pi^{-1}\pi(0)$ since π separates the closed orbits.

2. Let $\mathcal{O}(V)^G = k[f_1, \ldots, f_m]$ where the $f_i \in \mathcal{O}(V)^G$ are homogeneous of degrees ≥ 1. Then the quotient map is

$$\pi = (f_1, \ldots, f_m) \colon V \longrightarrow V/G \hookrightarrow k^m,$$

therefore

$$x \in V_u \Longleftrightarrow f_1(x) = \ldots = f_m(x) = 0 \Longleftrightarrow x \in V(f_1, \ldots, f_m),$$

the set of common zeros. This was HILBERTs definition of unstable ("Nullform" in the case where $G = SL_n$ and $V = R_d$, the space of homogeneous forms of degree d).

3. In particular V_u is a closed cone in V, the "nullcone", where "cone" means that $x \in V_u$ and $\lambda \in k^\times$ imply that $\lambda x \in V_u$.
4. x is semistable if and only if there is a homogeneous invariant $f \in \mathcal{O}(V)^G$ of degree ≥ 1 such that $f(x) \neq 0$. In particular $V_{ss} = V - V_u$ is an open cone in V.
5. The action is unstable $\Longleftrightarrow V_u = V \Longleftrightarrow 0$ is in the closure of every orbit $\Longleftrightarrow V/G$ has only one element $\Longleftrightarrow \mathcal{O}(V)^G = k$.
6. An example for 5 is $G = k^\times \cdot \mathbf{1}_V \subseteq GL(V)$ as well as any group between $k^\times \cdot \mathbf{1}_V$ and $GL(V)$.

Note (without proof) Let G be semisimple. Then

- Only very few G-modules have an unstable G-action.
- Only finitely many G-modules have a G-action that is not stable.

This was shown (at least in characteristic 0) by G. Elashvili [**?**].

Proposition 1 *Let G be reductive, V be a rational G-module, and $\pi \colon V \longrightarrow V/G$ be the good quotient. Then $V_s \subseteq V$ is open, and the restriction $\pi \colon V_s \longrightarrow \pi(V_s)$ is a geometric quotient, that is, its fibers are the orbits.*

Proof. Since V_s consists of closed orbits in V, the map π separates them. We have only to show that V_s is open. We may assume that the action is stable, that is $V_s \neq \emptyset$.

Let $m = \max\{\dim G{\cdot}x | x \in V\}$. Then the set $Z = \{z \in V \mid \dim G{\cdot}z < m\}$ is closed and G-stable. Hence $\pi Z \subset V/G$ is closed, and $U := (V/G) - \pi Z$ is open. Thus $\pi^{-1}U \subseteq V$ is open and G-stable and consists only of m-dimensional orbits. Claim: $\pi^{-1}U = V_s$.

Assume $x \in \pi^{-1}U$. Then $\pi x \notin \pi Z$. Hence the closed orbit that is contained in the closure of $G \cdot x$ has dimension at least m. But $\dim G \cdot x \leq m$, hence $G \cdot x$ itself is this closed orbit, thus $x \in V_s$.

Conversely if $x \in V_s$, then $\pi x \notin \pi Z$, hence $x \in \pi^{-1}U$. $\diamond$

3 One-Parameter Subgroups

A (multiplicative) one-parameter subgroup of an algebraic group G is (by abuse of notation) a homomorphism

$$\lambda\colon \mathbb{G}_m \longrightarrow G$$

of algebraic groups where $\mathbb{G}_m = k^\times$ is the multiplicative group. Let V be a rational G-module. If for $x \in V$ the morphism

$$k^\times \longrightarrow V, \quad t \mapsto \lambda(t) \cdot x,$$

extends to a morphism $\tilde{\lambda}\colon k \longrightarrow V$, then we use the notation (by another abuse)

$$\tilde{\lambda}(0) =: \lim_{t\to 0} \lambda(t) \cdot v.$$

(In algebraic geometry this kind of "limit" is often called specialization. For $k = \mathbb{C}$ this is a limit in the sense of analysis.)

Theorem 1 (The HILBERT-MUMFORD **criterion)** *Let G be a connected reductive algebraic group, $T \leq G$ be a maximal torus, V be a rational G-module, and $x \in V$. Then the following statements are equivalent:*

(i) *x is unstable, i. e. $0 \in \overline{G \cdot x}$.*

(ii) *The orbit $G \cdot x$ contains an element y that is unstable for T, i. e. $0 \in \overline{T \cdot y}$.*

(iii) *There is a one-parameter subgroup $\lambda\colon \mathbb{G}_m \longrightarrow G$ with $\lim_{t=0} \lambda(t) \cdot x = 0$.*

For the original proof see [1]. For a somewhat more elementary proof over $\mathbb{C}$ see [2]. We give a proof that uses well-known facts from the theory of algebraic groups.

Proof. "(iii) $\Longrightarrow$ (i)" is trivial.

"(i) $\Longrightarrow$ (ii)": We use

- An element of G is semisimple if and only if its conjugacy class meets T. In other words, $G_s = \bigcup_{g\in G} gTg^{-1}$, where G_s is the set of semisimple elements [**?**].
- G_s is an open dense subset of G [**?**].
- The induced morphism $G/G_x \longrightarrow G \cdot x$ is (purely inseparable and) a homeomorphism [**?**].
- The quotient map $G \longrightarrow G/G \cdot x$ is open [**?**].

These facts imply that $G_s \cdot x$ is a dense open subset of the orbit $G \cdot x$, and hence also of its closure $Y = \overline{G \cdot x}$. Thus

$$0 \in \overline{G_s \cdot x} = \overline{\bigcup_{g \in G} gTg^{-1} \cdot x} \subseteq \overline{\bigcup_{g \in G} gTg^{-1} \cdot x}$$

"(ii) $\Longrightarrow$ (ii)": $\diamond$

Corollary 1 *Under the same conditions x is properly stable if and only if for all one-parameter subgroups $\lambda : \mathbb{G}_m \longrightarrow G$, $\lambda(\mathbb{G}_m) \nsubseteq G_x$, the limit $\lim_{t=0} \lambda(t) \cdot x$ does not exist.*

Or, logically equivalent:

Corollary 2 *Under the same conditions x is not properly stable if and only if there exists a one-parameter subgroup $\lambda \colon \mathbb{G}_m \longrightarrow G$, $\lambda(\mathbb{G}_m) \nsubseteq G_x$, such that $\lim_{t=0} \lambda(t) \cdot x$ exists.*

4 Binary Forms

We consider the group $G = SL_2(k)$ of 2×2-matrices with determinant 1 over k. The matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

acts on the 2-dimensional vector space k^2 by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Denote the coordinate functions $k^2 \longrightarrow k$ by X and Y, where

$$X \begin{pmatrix} x \\ y \end{pmatrix} = x, \quad Y \begin{pmatrix} x \\ y \end{pmatrix} = y$$

for all $x, y \in k$. Since the inverse of g is

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

the contragredient action on the space of linear forms spanned by the coordinate functions X and Y is given by

$$\begin{aligned} X &\mapsto dX - bY, \\ Y &\mapsto -cX + aY. \end{aligned}$$

(In general a function $f\colon k^2 \longrightarrow k$ is transformed to $f \circ g^{-1}$.) This action extends to the polynomial ring $k[X,Y]$ as automorphisms. The homogeneous polynomials in $k[X,Y]$ of degree d (or "binary forms") form the SL_2-module

$$V = R_d = \{a_0X^d + a_1X^{d-1}Y + \cdots + a_dY^d \mid a_0, \ldots, a_d \in k\}.$$

Each $F \in R_d$ decomposes into a product of linear factors,

$$F = L_1^{m_1} \cdots L_r^{m_r}$$

where the $L_j \in R_1$ are pairwise different, $m_1 + \cdots + m_r = d$, and $m_1 \geq \ldots \geq m_r > 0$.

Assume $d \geq 1$, hence $r \geq 1$. Then a suitable matrix in SL_2 transforms F to the form

$$X^p \cdot g$$

where $g \in R_{d-p}$ has only linear factors of multiplicity $\leq p$ (or is constant for $p = d$). If moreover $r \geq 2$, then we may transform F even to

$$\tilde{F} = X^pY^q \cdot f \tag{1}$$

where $p \geq q$, $p+q \leq d$, and $f \in R_{d-p-q}$ has only linear factors of multiplicity $\leq q$. This transformation applies also for $r = 1$ if we allow $q = 0$. To summarize:

Lemma 1 *The SL_2-orbit of every $F \in R_d$ contains a form of type* (1) *where $p \geq q \geq 0$, and f has only linear factors of multiplicity $\leq q$. In particular*

$$\tilde{F} = \sum_{\nu=p}^{d-q} a_\nu X^\nu Y^{d-\nu}. \tag{2}$$

Theorem 2 *An element $F \in R_d$ is unstable for the action of SL_2 if and only if F has a linear factor of multiplicity $p > d/2$.*

Proof. We consider the one-parameter subgroup

$$\lambda\colon \mathbb{G}_m \longrightarrow SL_2, \quad \lambda(t) = \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & t \end{pmatrix}$$

and apply $\lambda(t)$ with $t \in k^\times$ to the form $\tilde{F}$ from equation (2):

$$\lambda(t) \cdot \tilde{F} = \sum_{\nu=p}^{d-q} a_\nu t^{2\nu-d} X^\nu Y^{d-\nu} = a_p t^{2p-d} X^p Y^{d-p} + \cdots + a_{d-q} t^{d-2q} X^{d-q} Y^q.$$

This has a specialization $\lim_{t\to 0} \lambda(t) \cdot \tilde{F}$ for $t = 0$ if and only if $2p - d \geq 0$, and

$$\lim_{t\to 0} \lambda(t) \cdot \tilde{F} = 0 \iff 2p - d > 0 \iff p > \frac{d}{2}.$$

This proves the if-part of the proposition.

For the converse we assume that $F \neq 0$ is unstable. Then by Theorem 1 there is a one-parameter subgroup λ with $\lim_{t\to 0} \lambda(t) \cdot F = 0$. The image $\lambda(\mathbb{G}_m)$ is nontrivial, hence a one-dimensional torus in SL_2, hence conjugated with the maximal torus

$$T = \left\{ \begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix} | s \in k^\times \right\},$$

$\lambda(\mathbb{G}_m) = gTg^{-1} = \tau_g T$ for some $g \in SL_2$ where τ_g is conjugation by g.

Let $\pi\colon SL_2 \longrightarrow k$ be the projection to the left upper coordinate. Then $\pi \circ \tau_g^{-1} \circ \lambda$ is an endomorphism of $\mathbb{G}_m$, hence of the form $\pi \circ \tau_g^{-1} \circ \lambda(t) = t^r$ for some $r \in \mathbb{Z}$, $r \neq 0$. Let us introduce the one-parameter subgroup

$$\rho\colon \mathbb{G}_m \longrightarrow T, \quad \rho(t) = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}.$$

Then $\tau_g^{-1} \circ \lambda(t) = \rho(t)$ for all $t \in k^\times$, hence $\lambda(t) = g\rho(t)g^{-1}$. For the polynomial $\bar{F} := g^{-1} \cdot F$ we have

$$\rho(t) \cdot \bar{F} = \rho(t)g^{-1} \cdot F = g^{-1}\lambda(t) \cdot F$$

hence $\lim_{t\to 0} \rho(t) \cdot \bar{F} = g^{-1} \cdot 0 = 0$. We write

$$\bar{F} = \sum_{\nu=0}^{d} b_\nu X^\nu Y^{d-\nu},$$

$$\rho(t) \cdot \bar{F} = \sum_{\nu=0}^{d} b_\nu t^{r(2d-\nu)} X^\nu Y^{d-\nu}.$$

In the case $r > 0$ the vanishing of the limit implies $b_\nu = 0$ for $d - 2\nu \leq 0$, $\nu \geq \frac{d}{2}$, thus

$$Y^p \mid \bar{F} \quad \text{with } p = \lceil \frac{d+1}{2} \rceil.$$

In the case $r < 0$ we likewise get

$$X^p \mid \bar{F} \quad \text{with } p = \lceil \frac{d+1}{2} \rceil.$$

In any case $\bar{F}$ has a linear factor of multiplicity $> d/2$, and so has F. $\diamond$

An equivalent formulation of Proposition 2 is:

Corollary 1 *An element $F \in R_d$ is semistable for the action of SL_2 if and only if all the linear factors of F have multiplicity $\leq d/2$.*

By a slight modification of the proof—looking for the existence of the limit rather than for its vanishing—we get that in the case $r > 0$ the existence of the limit implies $b_\nu = 0$ for $d - 2\nu < 0$, $\nu > \frac{d}{2}$, thus

$$Y^p \mid \bar{F} \quad \text{with } p = \lceil \frac{d}{2} \rceil.$$

Likewise for $r < 0$

$$X^p \mid \bar{F} \quad \text{with } p = \lceil \frac{d}{2} \rceil,$$

in any case a factor of multiplicity $\geq \frac{d}{2}$ if F is not properly stable. This proves the equivalence of (i) and (ii) in the following:

Corollary 2 *Assume $d \geq 3$, and $F \in R_d$. Then the following statements are equivalent:*

(i) *All linear factors of F have multiplicity $< d/2$.*

(ii) *F is properly stable for the action of SL_2.*

(iii) *F is stable for the action of SL_2.*

Proof. "(iii) $\Rightarrow$ (ii)": For $d \geq 3$ there exist properly stable points. Hence stable points must be properly stable. $\diamond$

References

[1] D. Mumford, J. Fogarty, F. Kirwan: *Geometric Invariant Theory* (3rd edition). Springer-Verlag, Berlin 1994.

[2] B. Sury: An elementary proof of the Hilbert-Mumford criterion. Electronic Journal of Linear Algebra: Vol. 7 (2000), Article 12. Available at: `http://repository.uwyo.edu/ela/vol7/iss1/12`