Extension of Conformal Maps to the Boundary

Klaus Pommerening
Fachbereich Mathematik der Johannes-Gutenberg-Universität
Mainz, Federal Republic of Germany

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The aim of this note is an elementary proof of the following theorem:

Theorem 1 *Let D be a bounded simply connected domain in the complex plane $\mathbb{C}$. Then the following statements are equivalent:*

(i) *Every conformal mapping of D onto the unit disk U extends to a homeomorphism of $\bar{D}$ onto $\bar{U}$.*

(ii) *The domain D is a Jordan domain.*

(iii) *Every boundary point of D is simple.*

Comments

1. A **Jordan domain** is a domain whose boundary is the support of a closed Jordan curve. Therefore the implication "(i) $\Longrightarrow$ (ii)" is an immediate consequence of the Riemann mapping theorem that gives a conformal mapping of D onto U. Extend it. The inverse mapping of the extension gives a homeomorphism of the unit circle onto the boundary ∂D.

2. The implication "(ii) $\Longrightarrow$ (i)" is the classical extension theorem conjectured by Osgood and proved by Caratheodory [1] and Osgood [6][7]. According to the Jordan Curve Theorem a Jordan domain is automatically bounded and simply connected. In the proof of the above theorem we make no use of the Jordan Curve Theorem. We prove "(ii) $\Longrightarrow$ (i)" via the chain "(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i)".

3. The implication "(iii) $\Longrightarrow$ (i)" is the version of the extension theorem given by Rudin [8]. A boundary point b of a domain D is **simple**, if every sequence in D with limit b can be connected by a curve $\gamma\colon [0,1] \longrightarrow \bar{D}$ such that $\gamma(1) = b$ and $\gamma(t) \in D$ for $0 \leq t < 1$. Equivalently we have: Given $\varepsilon > 0$ and a sequence (z_n) in D such that $\lim z_n = b$, almost all z_n are contained in the same connected component of $U_\varepsilon(b) \cap D$, where
$$U_\varepsilon(b) = \{z \in \mathbb{C} \mid |z - b| < \varepsilon\}.$$

Moreover we have a sufficient condition:

Lemma 1 *A boundary point b of D is simple, if for every $\varepsilon > 0$ there is an open neighborhood V of b such that V is contained in $U_\varepsilon(b)$ and $V \cap D$ is connected.*

Novinger [4] has given an elementary proof of "(iii) $\Longrightarrow$ (i)". Here we give a simpler proof. The idea is to consider the inverse conformal mapping $f : U \longrightarrow D$. So in the proof well-definition and injectivity interchange their roles.

Step 1

Extend f to a contimuous map $g\colon \bar{U} \longrightarrow \bar{D}$. This can be done as in the proof of [8, 14.19] using the following lemma.

Lemma 2 *Let (z_n) and (w_n) be sequences in U such that $\lim z_n = \lim w_n = a \in \partial U$. Let $\lim f(z_n) = b_1 \in \partial D$ and $\lim f(w_n) = b_2 \in \partial D$. Then $b_1 = b_2$.*

For the proof see [8, 14.18 (b)].

Step 2

Prove that g is injective on ∂D. This can be done as in [8, 14.18 (a)], but in view of step 1 we don't need Fatou's theorem on radial limits – the following well-known lemma is sufficient.

Lemma 3 *Let the function $g\colon \bar{U} \longrightarrow \mathbb{C}$ be continous in $\bar{U}$, analytic in U, and constant on a subarc of ∂D. Then g is constant.*

For a simple proof see [2].

The following simplification of step 2 is due to H. J. Fendrich (oral communication): Let a and b be distinct points of ∂D. The points a and b define two closed subarcs A and B of ∂D. Suppose $g(a) = g(b) = c$. We claim that g is constant on A or on B.

Assume the contrary. Then we find $x \in A$ and $y \in B$ such that $g(x) \neq c$ and $g(y) \neq c$. The image $g(C)$ of the chord $C = [x, y]$ is compact, and its distance ε from the point c is positive. Choose sequences $(z_n) \to a$ and $(w_n) \to b$ in U. The sequence $g(z_1), g(w_1), g(z_2), g(w_2), \ldots$ has limit c. Since c is a simple boundary point, ther is a curve γ_m in $U_\varepsilon(c) \cap D$ connecting $g(z_m)$ and $g(w_m)$ for each large m. The curve $g^{-1} \circ \gamma_m$ connects z_m and w_m, therefore it meets C for large m. But the support of γ_m is contained in $U_\varepsilon(c)$ and does not meet $g(C)$. This contradicts the assumption.

The remainder of the proof of "(iii) $\Longrightarrow$ (i)" follows by an easy compactness argument as in [8].

Finally we prove "(ii) $\Longrightarrow$ (iii)". We need a topological result that appears, with an elementary proof, in [3, Appendix to Chapter IX, (Ap. 3.2), and Exercise 2]:

Theorem 2 (Janiszewski) *Let A be a compact and B, a closed subset of the plane $\mathbb{C}$, a and b two distinct points of $\mathbb{C} - (A \cup B)$ such that neither A nor B separates a and b. Let $A \cap B$ be connected. Then $A \cup B$ does not separate a and b.*

Remark. A set A **separates** two points a and b, if a and b are contained in distinct connected components of $\mathbb{C} - A$.

Now assume (ii). Take a point $b \in \partial D$ and an arbitrary $\varepsilon > 0$. By the continuity of the boundary curve, b is contained in an open subarc J of $\partial D \cap U_\varepsilon(b)$. Therefore we have an open set V such that $V \subseteq U_\varepsilon(b)$ and $J = \partial D \cap V$. Since J is connected, we may suppose V connected. Let w and z be points of $V \cap D$. Then neither $\mathbb{C} - V$ nor ∂D separates w and z, and $\partial D \cap (\mathbb{C} - V)$ is the complementary arc of J, hence connected. By Janiszewski's theorem the set $(\mathbb{C} - V) \cup \partial D$ does not separate w and z. So w and z are in the same connected component of $V - \partial D$. We conclude that $V \cap D$ is connected. The statement (iii) follows by Lemma 1.

Some applications of the extension theorem are given in [5]. Here is another application:

Corollary 1 *Every bounded convex domain in the plane is a Jordan domain.*

Proof. A bounded convex domain is simply connected, and every boundary point is simple. ◇

References

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